

(8 pages)

Reg. No. : .....

Code No. : 5371

Sub. Code : ZMAM 23

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Second Semester

Mathematics — Core

ADVANCED CALCULUS

(For those who joined in July 2021 onwards)

Time : Three hours

Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. The value of  $\int_1^2 \sqrt{x} dx$  is

(a)  $\frac{4\sqrt{2}-2}{5}$

(b)  $\frac{4\sqrt{2}-2}{3}$

(c)  $\frac{4\sqrt{2}-1}{3}$

(d)  $\frac{\sqrt{2}-4}{3}$

2. Let  $D$  be the region between the line  $y = x$  and the parabola  $y = x^2$ . Let  $f(x, y) = xy^2$ . Then  $\iint_D f$  is

(a)  $\frac{1}{20}$

(b)  $\frac{1}{42}$

(c)  $\frac{1}{30}$

(d)  $\frac{1}{40}$

3. The image of the line  $x = 0$  under the

transformation  $S : \begin{cases} u = x + y \\ v = x - y \\ w = x^2 \end{cases}$  is

(a)  $u = v$

(b) the line  $u + v = 0$ , in the plane  $w = 0$

(c)  $u = 0, v = 0, w = 0$

(d) a circle

4. Let  $T$  be the linear transformation on  $R^2$  into  $R^2$  specified by the matrix  $\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$ . The image of the point  $(1, 2)$  is

(a)  $(0, 3)$

(b)  $(3, 0)$

(c)  $(0, -3)$

(d)  $(-3, 0)$

Page 2

Code No. : 5371





5. The Jacobian of the transformation  $T: \begin{cases} u = x \cos y \\ v = x \sin y \end{cases}$

is

- (a) 0 (b)  $y$   
(c)  $x$  (d) 1

6. If  $T: \begin{cases} u = \cos(x + y^2) \\ v = \sin(x + y^2) \end{cases}$  then at  $(x, y)$  the Jacobian of

$T$  is

- (a) 0  
(b)  $4y \sin(x + y^2) \cos(x + y^2)$   
(c)  $2y \sin(x + y^2) \cos(x + y^2)$   
(d) 2

7. If  $F$  is additive on  $A$ , the  $F(S_1 \cup S_2)$  is

- (a)  $F(S_1) + F(S_2)$   
(b)  $F(S_1) + F(S_2) + F(S_1 \cap S_2)$   
(c)  $F(S_1) + F(S_2) - F(S_1 \cap S_2)$   
(d)  $F(S_1) + F(S_2) - 2F(S_1 \cap S_2)$

8. The direction of the line through  $(1, 2, -1)$  towards  $(3, 1, 1)$  is

- (a)  $\left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$  (b)  $(2, -1, 2)$   
(c)  $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$  (d)  $\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$

Page 3

Code No. : 5371

9. If  $V = Ai + Bj + Ck$  then  $\text{curl}(V)$  is

- (a)  $(C_2 - B_3)i + (A_3 - C_1)j + (B_1 - A_2)k$   
(b)  $(C_2 - B_3)i + (A_2 - C_1)j + (B_2 - A_1)k$   
(c)  $(C_1 - B_1)i + (C_2 - B_2)j + (A_3 - B_1)k$   
(d)  $(C_2 - B_3)i - (A_3 - C_1)j + (B_1 - A_2)k$

10. If  $\alpha$  is a  $k$  form and  $\beta$  any differential form, then  $d(\alpha\beta)$  is

- (a)  $(d\alpha)\beta + (-1)^k \alpha(d\beta)$   
(b)  $(d\alpha)\beta + (d\beta)\alpha$   
(c)  $(d\alpha)\beta - \alpha(d\beta)$   
(d)  $(-1)^k (d\alpha)\beta + (-1)^k \alpha(d\beta)$

PART B — ( $5 \times 5 = 25$  marks)

Answer ALL questions choosing either (a) or (b).

11. (a) Let  $f$  and  $g$  be continuous and bounded on  $D$ .

Prove that  $\iint_D |f|$  exists and  $\left| \iint_D f \right| \leq \iint_D |f|$ .

Or

- (b) Show that for  $x > 0$

$$\int_0^{\pi/2} \log(\sin^2 \theta + x^2 \cos^2 \theta) d\theta = \pi \log\left(\frac{x+1}{2}\right).$$

Page 4

Code No. : 5371

[P.T.O.]





12. (a) Let  $T: \begin{cases} r = xy \\ s = 2x \\ t = -y \end{cases}$ ,  $S: \begin{cases} u = r - s \\ v = st \end{cases}$ .

Calculate the products  $ST$  and  $TS$ . Verify whether  $ST = TS$  or not.

Or

- (b) Define the differential of a transformation  $T$  compute the differential of

$$T: \begin{cases} u = x + 6y \\ v = 3xy \\ w = x^2 - 3y^2 \end{cases} \quad \text{at } (1, 1)$$

13. (a) Discuss the solution of the equations for  $u$  and  $v$

$$\begin{cases} x^2 - yu = 0 \\ xy + uv = 0 \end{cases}$$

Or

- (b) Let  $T$  be of class  $C'$  in an open region  $D$  and let  $E$  be a closed bounded subset of  $D$ . Let  $dT/p_0$  be the differential of  $T$  at a point  $p_0 \in E$ . Prove that

$$T(p_0 + \Delta p) = T(p_0) + dT/p_0(\Delta p) + R(\Delta p) \quad \text{where}$$

$$\lim_{\Delta p \rightarrow 0} \frac{|R(\Delta p)|}{|\Delta p|} = 0 \quad \text{uniformly for } p_0 \in E.$$

Page 5

Code No. : 5371

14. (a) If  $E$  is a closed bounded subset of  $\Omega$  at zero volume, prove that  $T(E)$  has zero volume.

Or

- (b) If  $\gamma_1$  and  $\gamma_2$  are smoothly equivalent curves, prove that  $L(\gamma_1) = L(\gamma_2)$ .

15. (a) If  $\omega$  is any differential form of class  $C''$ , prove that  $dd\omega = 0$ .

Or

- (b) State and prove the divergence theorem for the case of a cube.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If  $f$  is continuous on  $R$ , prove that  $\iint_R f$  exists.

Or

- (b) Let  $R$  be the rectangle described by  $a \leq x \leq b$ ,  $c \leq y \leq d$  and let  $f$  be continuous on  $R$ . Prove

$$\text{that } \iint_R f = \int_a^b dx \int_c^d f(x, y) dy.$$

Page 6

Code No. : 5371





17. (a) Let  $L$  be a linear transformation from  $R^n$  into  $R^m$  represented by the matrix  $[a_{ij}]$ . Prove that there is a constant  $B$  such that  $|L(p)| \leq B|p|$  for all points  $p$ . Also show that the number  $B$  is not the smallest number with this property.

Or

- (b) Let  $T$  be differentiable on an open set  $D$  and let  $S$  be differentiable on an open set containing  $T(D)$ . Prove that  $ST$  is differentiable on  $D$  and if  $p \in D$  and  $q = T(p)$ , then  $d(ST)_p = dS/dT_p$ .
18. (a) Let  $T$  be a transformation from  $R^n$  into  $R^n$  which is of class  $C'$  in an open set  $D$ , and suppose that  $J(p) \neq 0$  for each  $p \in D$ . Prove that  $T$  is locally 1-to-1 in  $D$ .

Or

- (b) Let  $T$  be of class  $C'$  on an open set  $D$  in  $n$  space, taking values in  $n$  space. Suppose that  $J(p) \neq 0$  for all  $p \in D$ . Prove that  $T(D)$  is an open set.

Page 7

Code No. : 5371

19. (a) Let  $F$  be an additive set function defined on  $\mathcal{G}$  and a.c. Suppose also that  $F$  is differentiable everywhere, and uniformly differentiable on compact sets, with the derivative a point function  $f$ . Prove that  $f$  is continuous everywhere and  $F(S) = \iint_S f$  holds for every rectangle  $S$ .

Or

- (b) Define a smooth curve. If  $\gamma$  is a smooth curve whose domain is the interval  $[a, b]$ . Prove that  $\gamma$  is rectifiable and  $L(\gamma)$  is given by the formula  $L(\gamma) = \int_a^b |r'(t)| dt$ .

20. (a) Prove that  $T^*(d\omega) = (d\omega)^* = d(w^*) = dT^*(\omega)$  when (i)  $\omega$  is a 0-form (ii)  $\omega$  is any 1-form.

Or

- (b) Give a proof of Stoke's theorem by reducing it to an application of Green's Theorem.

Page 8

Code No. : 5371

