(7 pages) **Reg. No. :** ....

Code No. : 8367 **Sub. Code : HMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2014.

Fourth Semester

Mathematics

FUNCTIONAL ANALYSIS

(For those who joined in July 2012 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Norm in the real space $\mathbb{R}$ is ——————
   (a) $|x|$ (b) x
   (c) $-x$ (d) $-|x|$

2. If a banach space B is reflexive then B^* is ———
   (a) Hausdorff (b) Compact
   (c) Connected (d) Reflexive

3. The only vector orthogonal to itself is ——————
   (a) Zero (b) Unit vector 1
   (c) -1 (d) None of these

4. If $x \perp y$ then $\|x-y\|^2 =$ ——————
   (a) $\|x\|^2 + \|y\|^2$ (b) $\|x\|^2 - \|y\|^2$
   (c) $\|x\|^2$ (d) $\|y\|^2$

5. If an operator T on H is self – adjoint then
   (a) T is not normal
   (b) T is unitary
   (c) T is positive
   (d) The imaginary part of T is zero

6. $(T_1T_2)^* =$ ——————
   (a) $T_1^* \, T_2^*$ (b) $T_2^* \, T_1^*$
   (c) $T_1T_2^*$ (d) $T_2T_1^*$

7. An operator T on H is singular iff ——————
   (a) $0 \in \sigma(T)$ (b) $1 \in \sigma(T)$
   (c) $-1 \in \sigma(T)$ (d) $\lambda \in \sigma(T)$

8. If T is normal then each M_i reduces ——————
   (a) T (b) M
   (c) T^* (d) M^*

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9. The boundary of S is a ———— of Z

(a) Superset (b) Subset

(c) Closed set (d) None

10. Which of the following is false?

(a) Any banach algebra which is a division algebra equals C

(b) If O is the only topological divisor of zero in A then $A = C$

(c) All topological divisor of zero in a banach algebra are divisors of zero

(d) Spectral radius of x is less than are equal to norm of x

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions choosing either (a) or (b).

11. (a) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, prove that there exists a functional f_0 in N^*, such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

Or

(b) Let T be a linear transformation of a normed linear space N into N'. Prove that T is continuous if and only if it is bounded.

12. (a) If M is a closed linear subspace of a Hilbert space H then prove that $H = M \oplus M^{\perp}$.

Or

(b) If M and N are closed linear subspace of a Hilbert space H such that $M \perp N$ prove that the linear subspace $M + N$ is also closed.

13. (a) If T is an operator on H for which $< Tx, x >= 0, \forall x$ then prove that $T = 0$.

Or

(b) If A is a positive operator on H, show that $I + A$ is non – singular, $I + T^*T$ and $I + TT^*$ are also non – singular for an arbitrary operator T on H.

14. (a) Show that an operator T on H is normal if and only if its adjoint T^* is a polynomial in T.

Or

(b) Let T be an operator on H prove that if $\lambda \in \sigma(T)$ and if p is any polynomial, then $p(\lambda) \in \sigma(p(T))$.

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15. (a) Prove $r(x) = \lim \|x^n\|^{1/n}$, where x is an element in a Banach Algebra.

Or

(b) If r is an element of a Banach algebra A with the property that $1 - xr$ is regular for every x, prove that r is in R.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions choosing either (a) or (b).

16. (a) Let B be Banach space such that $B = M \oplus N$, where M and N are closed linear subspaces of B. For $z \in B$ define $\|z\|' = \|x\| + \|y\|$, where $z = x + y, x \in M, y \in N$. Prove that $\| \|'$ is a norm on B and B is a Banach space with this new norm also.

Or

(b) In n is a fixed positive integer, the spaces $l_p^n (1 \le p \le \infty)$ consist of a single underlying linear space with different norms defined on it. Show that these norms are all equivalent to one another.

17. (a) State and prove Open Mapping Theorem.

Or

(b) If M is a proper closed linear subspace of a Hilbert space H, show that there exists a non – zero vector z_0 in H such that $z_0 \perp M$.

18. (a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in M, prove that $x - \sum < x, e_i > e_i \perp e_j$ for each j.

Or

(b) Let H be a Hilbert space and let f be an arbitrary functional in H^*. Prove that there exists a unique vector y in H such that $f(x) = < x, y >$ for every x in H. Hence prove that for an operator T on H there is an operator T^* on H such that $(Tx, y) = (x, T^* y)$ for all x, y.

19. (a) Prove that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

Or

(b) If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of H, prove that $P = P_1 + P_2 \cdots + P_n$ is a projection if and only if the P_i's are pairwise orthogonal; and in this case P is a projection on $M = M_1 + \cdots + M_n$.

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20. (a) Prove that the spectrum of an element of a Banach algebra is non – empty.

Or

(b) Show that the radical R of A is a proper closed two – sided ideal.

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