(8 pages) **Reg. No. :** ....

Code No. : 5385 **Sub. Code : ZMAM 43**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Fourth Semester

Mathematics – Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Which one of the following is not true?

   (a) Every complete normed linear space is a Banach space

   (b) The norm is a continuous function

   (c) If the linear transformation T is continuous then T is bounded

   (d) If M is a closed linear subspace of a normed linear space N, then N/M is a Banach space.

2. Let N and N' be normed linear spaces. An isometric isomorphism of N into N' is a one – to – one linear transformation T of N into N^1 such that

   (a) $\|T(x)\| \le \|x\|$ for every x in N

   (b) $\|T(x)\| = \|x\|$ for every x in N

   (c) $\|T(x)\| = 1$

   (d) $T(x) = T(y) \Rightarrow x = y$

3. If X is a compact Hausdorff space, then $\mathcal{C}(X)$ is reflexive if and only if

   (a) X is a finite set

   (b) X is a countable set

   (c) X is complete

   (d) X is a Banach space

4. Let T be a linear transformation of B into B'. The graph of T is a subset of

   (a) B (b) B'

   (c) $B \times B'$ (d) $B' \times B$

5. The parallelogram law states that

(a) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 - 2\|y\|^2$

(b) $\|x+y\|^2 - 2\|x\|^2 = 2\|y\|^2 - \|x-y\|^2$

(c) $\|x+y\|^2 - \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(d) $\|x+y\|^2 + 2\|x\|^2 = \|x-y\|^2 + 2\|y\|^2$

6. If S is a non-empty subset of a Hilbert space then $S^{\perp\perp\perp}$ is

(a) S (b) $S^{\perp}$

(c) ϕ (d) $S^{\perp\perp}$

7. Which one of the property is not the property of the adjoint operation $T \to T^*$ on $\mathcal{B}(H)$

(a) $(T_1 + T_2)^* = T_2^* + T_1^*$

(b) $(T_1 T_2)^* = T_1^* T_2^*$

(c) $T^{**} = T$

(d) $\|T^*T\| = \|T\|^2$

8. A self – adjoint operator A is said to be positive if

(a) $(Ax, Ax) \geq 0$ for all x

(b) (Ax, x) is real for all x

(c) $(Ax, x) \geq 0$ for all x

(d) $\|A^2\| = \|A\|^2$

9. An operator T on H is self adjoint if and only if

(a) $T + T^* = 0$

(b) (Tx, x) is real for all x

(c) $TT^* = T^*T$

(d) $\|T^* x\| = \|Tx\|$ for every x

10. If P is the projection on M then $I - P$ is the projection on

(a) M

(b) $M^{\perp}$

(c) $I - M$

(d) $M - M$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define a Banach space with two examples.

Or

(b) If M is a closed linear subspace of a normed linear space N and x_0 is vector not in M, prove that there exists a functional f_0 in N^* such that $f_0(m) = 0$ and $f_0(x_0) \neq 0$.

12. (a) If N is a normed linear space, prove that the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak* topology.

Or

(b) State and prove the closed graph theorem.

13. (a) Prove that the inner product in a Hilbert space is jointly continuous.

Or

(b) Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Prove that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

14. (a) Let $\{e_1, e_2 .., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, prove that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ and $x - \sum_{1}^{n} (x, e_i)e_i \perp e_j$ for each j.

Or

(b) Show that an orthonormal set in a Hilbert space is linearly independent.

15. (a) If T is an operator on H for which $(Tx, x) = 0$ for all x, prove that $T = 0$.

Or

(b) Prove that an operator T on H is unitary $\Leftrightarrow$ it is an isometric isomorphism of H onto itself.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) Let M be a closed linear subspace of a normed linear space N. Prove that N/M is a normed linear space. If N is a Banach space, prove that N/M is also a Banach space.

Or

(b) If N' is a Banach space, prove that the normed linear space (N, N') is also a Banach space.

17. (a) Define a reflexive space with an example. If B is a Banach space, prove that B is reflexive $\Leftrightarrow B^*$ is reflexive.

Or

(b) Let B and B' be Banach spaces. Let T be a continuous linear transformation of B onto B'. Prove that the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.

18. (a) State and prove the Banach – Steinhaus theorem.

Or

(b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

19. (a) State and prove Bessel's inequality.

Or

(b) Let H be a Hilbert space and let f be an arbitrary functional is H^*. Prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

20. (a) If T is an operator on H, prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute

Or

(b) If P is a projection on H with range M and null space N, prove that $M \perp N \Leftrightarrow P$ is self adjoint. Also prove that $N = M^{\perp}$ in this case.
