(7 Pages)

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M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2022

Second Semester

Mathematics — Core

ALGEBRA — II

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. A homomorphism of R into R' is said to be an isomorphism if it is a ———— mapping.

   (a) onto (b) one to one

   (c) isomorphic (d) none of these

2. Let D be an integral domain, $a, b \in D$ suppose that $a^n = b^n$ and $a^m = b^m$ for two relatively prime positive integers m and n then a ———— b.

   (a) $>$ (b) $<$

   (c) $=$ (d) $\neq$

3. Let R be a commutative ring with unit element. An element $a \in R$ is a unit in R if there exists an element $b \in R$ such that $ab =$ ————

   (a) 1 (b) 0

   (c) ∞ (d) -1

4. If R is an Euclidean ring and $a, b \in R$. If $b \neq 0$ is not a unit in R then ————

   (a) $d(a) < d(ab)$ (b) $d(a) > d(ab)$

   (c) $d(a) = d(ab)$ (d) none of these

5. If $f(x) = a_0 + a_1 x + \ldots + a_n x^n \neq 0$ and $a_n \neq 0$ then the degree of $f(x)$, written as $\deg f(x)$, is ————

   (a) $n-1$ (b) $n+1$

   (c) 1 (d) n

6. If $f(x)$ and $g(x)$ are two polynomials, then ————

(a) $\deg(f(x)g(x)) \leq \deg f(x)$, $g(x) \neq 0$

(b) $\deg(f(x)g(x)) \geq \deg f(x)$, $g(x) \neq 0$

(c) $\deg(f(x)g(x)) = \deg f(x) \cdot \deg g(x)$, $g(x) \neq 0$

(d) $\deg(f(x)g(x)) = \deg f(x) - \deg g(x)$, $g(x) \neq 0$

7. The only idempotent element in rad R is ————

(a) 1 (b) 2

(c) 3 (d) 0

8. The ring Z of integers is ————

(a) prime radical (b) semi-simple

(c) prime ideal (d) none of these

9. For any ring R, $R/\text{rad } R$ is ———— to a sub direct sum of integral domains.

(a) monomorphic (b) isomorphic

(c) homomorphic (d) automorphic

10. If a and b are elements of $\Sigma \oplus R_i$ such that $\pi_i(a) = \pi_i(b)$ for each index I, then a ———— b.

(a) $=$ (b) $\neq$

(c) $>$ (d) $<$

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If ϕ is a homomorphism of R into R' with Kernel $I(\phi)$, then prove the following.

(i) $I(\phi)$ is a subgroup of R under addition.

(ii) If $a \in I(\phi)$ and $r \in R$ then both ar are ra are in $I(\phi)$.

Or

(b) If U is an ideal of R and $1 \in U$, prove that $U = R$.

12. (a) Prove that a Euclidean ring posses a unit element.

Or

(b) State and prove Fermat theorem.

13. (a) If $f(x), g(x)$ are two nonzero elements of $f[x]$, then prove that $\deg(f(x), g(x)) = \deg f(x) + \deg g(x)$.

Or

(b) State and prove Gauss' Lemma.

14. (a) Prove that an element a is invertible in the ring R iff the $coset\, a + rad\, R$ is invertible in the quotient ring $R / rad\, R$.

Or

(b) Prove that for any ring R, rad R is the smallest ideal I of R such that the quotient ring R/I is semi-simple (in other words, if R/I is a semi-simple ring, then rad $R \subseteq I$).

15. (a) Prove that an element a of the ring R is quasi-regular iff there exists some $b \in R$ such that $a + b - ab = 0$. The elements b satisfying this equation is called a quasi-inverse of a.

Or

(b) Prove that a ring R is isomorphic to a sub-direct sum of fields iff for each nonzero ideal I of R, there exists an ideal $J \neq R$ such that $I + J = R$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let R be a commutative ring with unit element whose only ideals are (0) and R itself Then, prove that R is a field.

Or

(b) Prove that every integral domain can be imbedded in a field.

17. (a) Prove that $J[i]$ is a Euclidean ring.

Or

(b) State and prove Unique Factorization theorem.

18. (a) Prove that the ideal $A = p(x)$ in $F[x]$ is a maximal ideal iff $p(x)$ is irreducible over F.

Or

(b) Prove that if R is a unique factorization domain and if $p(x)$ is a primitive polynomial in $R[x]$, then it can be factored in a unique way as the product of irreducible elements in $R[x]$.

19. (a) Prove that for any ring R, the quotient ring $R/rad\, R$ is semi-simple; that is, $rad(R/rad\, R) = \{0\}$.

Or

(b) Prove that a ring R is a primary ring iff R has a minimal prime ideal which contains all zero divisors.

20. (a) Define the J-Radical $J(R)$ and J-semi simple ring and prove that the ring of even integers is J-Semi simple.

Or

(b) Prove that for any ring R, the J-radical $J(R)$ is an ideal of R.

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