(8 pages) **Reg. No. :** ....

Code No. : 8522 **Sub. Code : HMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2013.

Second Semester

Mathematics

ALGEBRA — II

(For those who joined in July 2012 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let V is an algebra with unit element over F. Then which of the following is true

   (a) V is isomorphic to A(V)

   (b) V is isomorphic to a sub algebra of A(V)

   (c) V = A(V)

   (d) None of these

2. Let 1, 2, 3 are the characteristic roots of $T \in A(V)$. Then the characteristic roots of $T^2 + 2T + 3I$ is

   (a) 1, 2, 3 (b) 6, 11, 18

   (c) 3, 8, 15 (d) None of these

3. Let $T \in A(V)$ be nilpotent. Then $\alpha_0 + \alpha_1 T + \ldots.. + \alpha_m T^m$ is invertible if

   (a) $\alpha_0 = 0$

   (b) $\alpha_0 \neq 0$

   (c) $\alpha_0 = 1$

   (d) α_0 can be any value but $\alpha_m = 1$

4. Let T be the identity linear transformation on a finite dimensional vector space V of dimension n. Then the number of non zero entries

   (a) n (b) n^2

   (c) $n^2 - n$ (d) $n^2 + n$

5. Let $T \in A(V)$ be unitary. Which of the following could be the characteristic root of T

   (a) $1 + i\sqrt{3}$ (b) $\frac{1 - i\sqrt{3}}{2}$

   (c) $\frac{1}{2}$ (d) $\frac{i}{2}$

6. Let Q be a real orthogonal matrix. Then determinant of Q is

(a) 1 (b) -1

(c) Either (a) or (b) (d) Zero

7. What is the degree of $\sqrt{2}+\sqrt[3]{5}$ over Q?

(a) 2 (b) 3

(c) 5 (d) 6

8. Let F be the fields of rational numbers and let $f(x)=x^4+x^2+1\in F(x)$. If E is the splitting field of $f(x)$ then $[E:F]$ is

(a) 2 (b) 4

(c) 4 ! (d) None of these

9. Let F be a field with 121 elements. Then the number of non zero roots of $x^{121}-x\in F[x]$ in F is

(a) 11 (b) 121

(c) 120 (d) None of these

10. Let F be a field of order p^n. Then the number of automorphisms on F is

(a) p (b) n

(c) p^n (d) $\varphi(n)$

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let A be an algebra, with unit element over F, and suppose that A is of dimension m over F. Then prove that every element in A satisfies some nontrivial polynomial in $F[x]$ of degree atmost m.

Or

(b) If V is finite dimensional over F, then prove that $T\in A(V)$ is singular if and only if there exists a $v\neq 0$ in V such that $vT=0$.

12. (a) Let V be a subspace spaned by $v, vT, vT^2, \ldots\ldots, vT^{n_1-1}$. If $u\in V$ is such that $uT^{n_1-k}=0$ where $0<k\leq n_1$ then prove that $u=u_0T^k$ for some $u_0\in V$.

Or

(b) If $v=v_1\oplus v_2\oplus \ldots\oplus v_k$ where each subspace v_i is of dimension n_i and is invariant under T, an elements of $A(V)$ then prove that a basis of V can be found so that the matrix of T in this basis is of the form $\begin{pmatrix} A_1 & 0\ldots\ldots & 0 \\ 0 & A_2\ldots\ldots & 0 \\ 0 & 0\ldots\ldots & A_K \end{pmatrix}$ where each Ai is an $n_i\times n_i$ matrix and is the matrix of the linear transformation induced by T and v_i.

13. (a) Prove that the normal transformation N is Hermitian if and only if its characteristic roots are real.

Or

(b) If N is normal and AN = NA then prove that $AN^* = N^* A$.

14. (a) If a, b in K are algebraic over F then prove that $a \pm b$, ab and $\frac{a}{b}$ (if $b \neq 0$) are all algebraic over F.

Or

(b) If a is algebraic over F then prove that two minimal, monic polynomials for a over F are equal.

15. (a) If $f(x), g(x) \in f(x)$ prove that $(f(x)+g(x))' = f'(x)+g'(x)$ and $(\alpha f(x))' = \alpha f'(x)$.

Or

(b) Find primitive roots of 17.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that $S \in A(V)$ is regular if and only if whenever $v_1 \ldots\ldots v_n \in V$ are linearly independent then $v_1 S, v_2 S \ldots . v_n S$ are also linearly independent.

Or

(b) If V is two dimensional over a field F prove that every elements in $A(V)$ satisfied a polynomial of degree 2 over F.

17. (a) Let $T \in A(V)$ be nil potent. Prove that there exists subspaces v_1 and w of v, invariant under T, such that $v = v_1 \oplus W$.

Or

(b) Prove that two nil potent linear transformation are similar if and only if they have same invarients.

18. (a) If F is a field of characteristic 0 and if $T \in A_F(v)$ is such that $t_v T^i = 0$ for all $i \geq 1$ then prove that T is nilpotent.

Or

(b) (i) If $A, B \in F_n$ then prove that $t_r(AB) = t_r(BA)$. Also prove that if A is invertible then $t_r\left(A \subset A^{-1}\right) = t_r C$.

(ii) If $T \in A(V)$ then prove that $t_r T$ is the sum of the characteristic roots of T.

19. (a) Prove that the elements $a \in k$ is algebraic over F of degree n then prove that $[F(a) : F] = n$.

Or

(b) If $f(x)$ is a polynomial of degree n over a field F then prove that $f(x)$ have at most n roots in any extension field. Also prove that there is an extension E of F of degree at most $n!$ in which $f(x)$ has n roots.

20. (a) Let F be a finite field with q elements and suppose that $F \subset K$ where K is also a finite field. Then prove that K has q^n elements where $n = [K : F]$ and hence deduce that any finite field has p^m elements where the prime number p is the characteristic of that field.

Or

(b) Prove that for every prime number p and every positive integer m there is a unique field having p^m elements.
