(6 pages) **Reg. No. :** ....

Code No. : 6364 **Sub. Code : ZMAM 11**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2022.

First Semester

Mathematics — Core

ALGEBRA — I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. A homomorphism ϕ from G into $\overline{G}$ is said to be an isomorphism if ϕ is ————
   (a) one to one (b) unto
   (c) not one to one (d) bijective

2. Every subgroup of an abelian group is ————
   (a) right coset (b) last coset
   (c) normal (d) not normal

3. In a group $b^5 = e$ and $aba^{-1} = a^2$ for some $a, b \in G$. The order of a is ————
   (a) 5 (b) 10
   (c) 0 (d) divisor of 10

4. Let G be a group and ϕ an automorphism of G. If $a \in G$ is of order $o(a) > 0$, then $o(\phi(a)) =$ ————
   (a) 0 (b) 1
   (c) $o(a)$ (d) ∞

5. If $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ then $\alpha\beta =$ ————
   (a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
   (c) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

6. If $o(G) = p^2$ where p is a prime number then G is ————
   (a) normal (b) left coset
   (c) right coset (d) abelian

7. The value of $9c_2$ is ————
   (a) 18 (b) 8
   (c) 32 (d) 36

8. The number of p-sylow subgroups in G, for a given prime is of the form ———

(a) $1+kp$ (b) $1-kp$

(c) kp (d) $\frac{1+k}{p}$

9. If $\phi \neq 1 \in G$ where G is an abelian group then $\sum_{g\in G}\phi(g) =$ ———

(a) 1 (b) 2

(c) ∞ (d) 0

10. The number of non-isomorphic abelian groups of order p^n, p an prime, equals the number of partitions of ———.

(a) $\frac{n}{2}$ (b) $n!$

(c) n (d) $n-1$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If G is a finite group and N is a normal subgroup of G, then prove that $o(G/N) = o(G)/o(N)$.

Or

(b) If ϕ is a homomorphism of G into $\overline{G}$, then prove that :

(i) $\phi(e) = \overline{e}$, the unit element of $\overline{G}$.

(ii) $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$

12. (a) Show that $\mathcal{I}(G) \approx G/Z$, where $\mathcal{I}(G)$ is the group of inner automorphisms of G, and Z is the center of G.

Or

(b) If H is a subgroup of G show that for every $g \in G$, gHg^{-1} is a subgroup of G.

13. (a) Prove that $N(a)$ is a subgroup of G.

Or

(b) If $o(G) = p^n$ where p is a prime number, then prove that $Z(G) \neq (e)$.

14. (a) Prove that $n(k) = 1 + p + \ldots + p^{k-1}$.

Or

(b) If $p^m \mid o(G), p^{m+1} \nmid o(G)$, then prove that G has a subgroup of order p^m.

15. (a) Let G be a group and suppose that G is the integral direct production of $N_1, \ldots, N_n$. Let $T = N_1 \times N_2 \times \ldots \times N_n$. Then prove that G and T are isomorphic.

Or

(b) If G and G' are isomorphic abelian groups, then prove that for every integer s, $G(s)$, and $G'(s)$ are isomorphic.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

16. (a) State and prove Sylow's theorem for Abelian groups.

Or

(b) Let ϕ be a homomorphism of G onto $\overline{G}$ with kernel K, and let $\overline{N}$ be a normal subgroup of $\overline{G}, N = \{x \in G \mid \phi(x) \in \overline{N}\}$. Then prove that $G/N \approx \overline{G}/\overline{N}$. Equivalently. $G/N \approx (G/K)/(N/K)$.

17. (a) If G *is* a group, then prove that $\mathcal{A}(G)$, the set of automorphisms of G, is also a group.

Or

(b) Let G be a finite group, T an automorphism of G with the property that $xT = x$ iff $x = e$. Suppose further that $T^2 = 1$ prove that G must be abelian.

18. (a) State and prove Cauchy theorem.

Or

(b) Prove : $o(G) = \sum \frac{o(G)}{o(N(a))}$ where this sum runs over one element a in each conjugate class.

19. (a) State and prove Sylow theorem.

Or

(b) Prove that S_{p^k} has a p-sylow subgroup.

20. (a) Let G be an abelian group of order p^n, p a prime. Suppose that $G = A_1 \times A_2 \times \ldots \times A_k$, where each $A_i = (a_i)$ is cyclic of order p^{n_i}, and $n_1 \geq n_2 \geq \ldots \geq n_k > 0$. If m is an integer such that $n_t > m \geq n_{t+1}$ then prove that $G(p^m) = B_1 \times \ldots \times B_t \times A_{t+1} \times \ldots \times A_k$ where B_i is cyclic of order p^m, generated by $a_i^{p^{n_i - m}}$, for $i \leq t$. The order of $G(p^m)$ is p^u, where $u = mt\sum_{i=t+1}^{k} n_i$.

Or

(b) Show that the two abelian groups of order p^n are isomorphic iff they have the same invariants.
