(7 pages) **Reg. No. :**

Code No. : 9069 **Sub. Code : PMAM 11**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2017.

First Semester

Mathematics

ALGEBRA — I

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

SECTION A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. If ϕ is a homomorphism then $\phi(ab) =$ ————

   (a) $\phi(a)\phi(b)$

   (b) $\phi(a/b)$

   (c) $\phi(a) - \phi(b)$

   (d) None of these

2. If G is a group and H is a subgroup of index 2 in G then

   (a) H is a normal subgroup of G

   (b) H is an abelian subgroup of G

   (c) $aHa^{-1} \neq H$

   (d) None of these

3. Let G be a group of order 99 and let H be a subgroup of order 11. Then H contains a normal subgroup $N \neq (e)$ or order

   (a) 3 or 5 (b) 3 or 6

   (c) 11 (d) 9

4. If G is abelian of order $O(G)$ and $p^{\alpha} \mid O(G)$, $p^{\alpha+1} \nmid O(G)$, then

   (a) there is a subgroup of G of order p^{α}

   (b) there is a unique subgroup of G of order p^{α}

   (c) there must be a subgroup of G of order $> p^{2}$

   (d) none of these

5. Let A_n be the set of all even permutations in S_n. Then $O(A_n) =$ ————.

   (a) $\underline{|n}$ (b) $\frac{\underline{|n}}{2}$

   (c) n (d) $\frac{n}{2}$

6. Let $a \in Z$, centre of G. Then

(a) $N(a) = G$ (b) $N(a) \neq G$

(c) $O(Z) = p^n$ (d) none of these

7. S_{p^k} has a ———— sylow subgroups.

(a) p^k (b) p

(c) k (d) none

8. Let G be a group of order 72. Then

(a) G must have a non trivial normal subgroup

(b) G is simple

(c) G is not having any normal subgroup

(d) None of these

9. Every finite abelian group is the direct product of ————.

(a) group (b) abelian group

(c) cyclic group (d) none of these

10. The number of non-isomorphic abelian groups of order p^n, p prime, equals

(a) $p(n)$ (b) p^3

(c) p (d) none

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If H and K are finite subgroups of G of order $O(H)$ and $O(K)$, respectively, then $O(HK) = \dfrac{O(H)O(K)}{O(H \cap K)}$. Prove this.

Or

(b) If ϕ is a homomorphism of G onto $\overline{G}$ with Kernel K, then the set of all inverse images of $\overline{g} \in \overline{G}$ under ϕ in G is given by Kx where x is any particular inverse image of $\overline{g}$ in $\overline{G}$. Prove this.

12. (a) Prove that an automorphism group of an infinite cyclic group is a group of order 2.

Or

(b) If G is a group and N is a normal subgroup of G such that both N and G/N are solvable, prove that G is solvable.

13. (a) If G is a finite group, then show that $C_a = \dfrac{O(G)}{O(N(G))}$ where C_a is the number of elements conjugate to a in G.

Or

(b) If $O(G) = p^n$, where p is a prime number them show that $Z(G) \neq (e)$.

14. (a) Let G be a group and suppose that G is a subgroup of a finite group M. Suppose further that M has a P-sylow subgroup Q. Then show that G has a P-sylow subgroup P and $P = G \cap xQx^{-1}$

Or

(b) Show that the number of P-sylow subgroups in G equals index of $N(P)$ in G.

15. (a) Suppose G is the internal direct product of $N_1, N_2, \ldots, N_n$. Then for $i \neq j, N_i \cap N_j = \{e\}$ and if $a \in N_i$, $b \in N_j$ then $ab = ba$.

Or

(b) If G and G' are isomorphic abelian groups then show that for every integer s, $G(s)$ and $G'(s)$ are isomorphic.

SECTION C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) State and prove Sylow's theorem for abelian groups.

Or

(b) Let ϕ be a homomorphism of G onto $\overline{G}$ with Kernel K. Let $\overline{N}$ be a normal subgroup of $\overline{G}$, $N = \{x \in G / \phi(x) \in \overline{N}\}$. Then show that $G / N \cong \overline{G} / \overline{N}$.

17. (a) (i) Let G be a group. Let $\vartheta(G)$ denotes the set of all inner automorphisms of G. Prove that $\vartheta(G)$ is a subgroup of $A(G)$, set of all automorphisms of G.

(ii) Prove that $\vartheta(G)$ is isomorphic to G / Z where Z is the centre of the group G.

Or

(b) State and prove Cayley's theorem.

18. (a) Prove that the number of conjugate class in S_n is the number of partition of n.

Or

(b) State and prove Cauchy's theorem for general group.

19. (a) Let G be a group and let A, B be any two subgroups of G. If $x, y \in G$ then $x \sim y$ if $y = axb$ for some $a \in A$, $b \in B$. Then prove that

(i) $\sim$ is an equivalence relation on G.

(ii) The equivalent class of $x \in G$ is the set $AxB = \{axb / a \in A, b \in B\}$.

(iii) $O(AxB) = \dfrac{O(A)\,O(B)}{O(A \cap x B x^{-1})}$.

Or

(b) Prove that the number of p-sylow subgroups in G equals $\frac{O(G)}{O(N(p))}$, where p is any p-sylow subgroup of G.

20. (a) Let G be group and suppose that G is the internal direct products of $N_1, N_2, \ldots, N_n$, let $T = N_1 \times N_2 \times \ldots \times N_n$. Then prove that G and T are isomorphis.

Or

(b) Two abelian groups of order p^n are isomorphic iff they have the same invariants. Prove this.