(8 pages) **Reg. No. :** ....

Code No. : 7752 **Sub. Code : WMAM 12**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2023.

First Semester

Mathematics — Core

REAL ANALYSIS — I

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. If there exists a positive number M such that $\sum_{k=1}^{n}|\Delta f_k| \leq M$ for all partitions of $[a, b]$, then f is said to be of ———— variation on $[a, b]$.

(a) equal (b) less than

(c) bounded (d) unbounded

2. The total variation $V_f(a, b) = 0$ if and only if f is ———— on $[a, b]$.

(a) constant (b) variable

(c) equal (d) ten

3. A series $\sum a_n$ is called ———— convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

(a) absolutely (b) conditionally

(c) equally (d) unequally

4. The value of $\int_a^a f dx$ is ————

(a) 1 (b) ∞

(c) 0 (d) fx

5. Name the condition that f satisfies on $[a, b]$ that if for every $\epsilon > 0$, there exists a partition P_ϵ such that P finer than P_ϵ implies $0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ with respect to α.

(a) Riemann's (b) Cauchy

(c) Euler (d) Newton's

6. If $f \in R(\alpha)$ and $g \in R(\alpha)$, where $\alpha \uparrow$ on $[a, b]$ then the product $fg \in$ ————

(a) $R(\alpha)^2$ (b) $R(\alpha).R(\alpha)$

(c) $2R(\alpha)$ (d) $R(\alpha)$

7. One of the sufficient condition for the existence of the Riemann integral $\int_a^b f(x)\, dx$ is ———— on $[a, b]$.

(a) f is of bounded variation

(b) f is not continuous

(c) f unbounded

(d) f is of unbounded variation

8. If α be continuous and $f \nearrow$ on $[a, b]$, then there exists a point x_0 in $[a, b]$ such that $\int_a^b f(x)\, d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) +$ ————

(a) $f(a)\int_{x_0}^{b} d\alpha(x)$ (b) $f(b)\int_{x_0}^{b} d\alpha(x)$

(c) $\int_b^{x_0} f(b) d\alpha(x)$ (d) $\int_{x_0}^{b} d\alpha(x)$

9. For $f \in R$ and α a continuous function on $[a, b]$ whose derivative α' is Riemann integrable on $[a, b]$ then $\int_a^b f(x) d\alpha(x)$ ———— $\int_a^b f(x)\alpha'(x)\, dx$.

(a) = (b) ≠

(c) < (d) >

10. The double series is said to be ———— to the sum a if $\lim_{p, q \to 0} S(p, q) = a$.

(a) converge (b) diverge

(c) oscillate (d) all

11. The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is ————

(a) 0 (b) 1

(c) ∞ (d) 2

12. The series $\sum (-1)^{n+1}.n$ is ————

(a) $(C,1)$ summable (b) not $(C,1)$ summable

(c) Summable (d) not summable

13. A sequence $\{f_n\}$ is said to be ———— on S if there exists a constant $M > 0$ such that $|f_n(x)| \le M$ for all x in S and all n.

(a) Limit (b) Converge

(c) Uniformly bounded (d) Continuous

14. A sequence of functions $\{f_n\}$ is said to be ———— on T if $\{f_n\}$ is pointwise convergent and uniformly bounded on T.

(a) integrable

(b) converge

(c) boundedly convergent

(d) divergent

15. For the function $f_n(x) = x^n$ in $0 \le x \le 1$ which is continuous with discontinuous limit, the convergence is ———— on [0, 1].

(a) uniform (b) not uniform

(c) both (a) and (b) (d) cannot say

PART B — ($5 \times 4 = 20$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If f is monotonic on $[a, b]$, then prove that the set of all discontinuities of f is countable.

Or

(b) Assume that f and g are each of bounded variation on $[a, b]$. Then prove that their sum and product are also of bounded variation.

17. (a) Assume that $\alpha \nearrow$ on $[a, b]$ If $f \in R(\alpha)$ on $[a, b]$ then prove that $f^2 \in R(\alpha)$ on $[a, b]$.

Or

(b) If $f, g \in R(\alpha)$ on $[a, b]$ then show that $C_1 f + C_2 g \in R(\alpha)$ on $[a, b]$ for any two constants C_1 and C_2 and also $\int_a^b (C_1 f + C_2 g) d\alpha = C_1 \int_a^b f d\alpha + C_2 \int_a^b g d\alpha$.

18. (a) If f is continuous on $[a, b]$ and if α is of bounded variation on $[a, b]$ then show that $f \in R(\alpha)$ on $[a, b]$.

Or

(b) State and prove second mean value theorem for Riemann integrals.

19. (a) Prove that if a series is convergent with sum S, then it is also $(C, 1)$ summable with Cesaro sum S.

Or

(b) Let $a_n \geq 0$ then prove that the product $\prod(1 - a_n)$ converges if and only if the series $\sum a_n$ converges.

20. (a) State and prove Dirichlet's test for uniform convergence.

Or

(b) Assume $f_n \to f$ uniformly on S. If each f_n is continuous at a point C of S then prove that the limit function f is also continuous at C.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b)

21. (a) Let f be of bounded variation on $[a, b]$. If $x \in (a, b]$, let $V(x) = V_f(a, x)$ and put $V(a) = 0$. Then prove that every point of continuity of f is also a point of continuity of V prove converge also.

Or

(b) State and prove Riemann's theorem on conditionally convergent series.

22. (a) Assume that $\alpha \nearrow$ on $[a, b]$. Then prove that the following statements are equivalent.

(i) $f \in R(\alpha)$ on $[a, b]$

(ii) f satisfies Riemann's condition with respect to α on $[a, b]$

(iii) $\underline{I}(f,\alpha) = \overline{I}(f, \alpha)$.

Or

(b) Let $f \in R(\alpha)$ on $[a, b]$ and let g be a strictly monotonic continuous function defined on an interval S having end points c and d. Assume that $a = g(c), b = g(d)$. Let h and β be the composite functions defined as $h(x) = f[g(x)]$, $\beta(x) = \alpha[g(x)]$ if $x \in S$. Then prove that $h \in R(\beta)$ on S and $\int_a^b f d\alpha = \int_c^d h\, d\beta$

23. (a) State and prove theorem on change of variable in Riemann integral.

Or

(b) Assume that α is of bounded variation on $[a, b]$. Let $V(x)$ denote the total variation of α on $[a, x]$ if $a < x \leq b$ and $V(a) = 0$. Let f be defined and bounded on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$ then prove that $f \in R(V)$ on $[a,b]$.

24. (a) Write Bernstein's theorem and prove it.

Or

(b) State and prove Merten's theorem.

25. (a) State and prove three examples of sequences of real valued functions.

Or

(b) State and prove the theorem on Cauchy condition for uniform convergence.
