(7 pages) **Reg. No. :** ....

Code No. : 5369 **Sub. Code : ZMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION,
APRIL 2023

Second Semester

Mathematics – Core

ALGEBRA – II

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If R is a commutative ring and $a \in R$, the $aR = \{ar \mid r \in R\}$ is a ———— ideal of R

   (a) Right ideal (b) Left ideal

   (c) Two-sided ideal (d) None of the above

2. The number of ideals of the ring of rational numbers is ————

   (a) 2 (b) 1

   (c) 0 (d) none of the above

3. The gcd of $3+4i$ and $4+3i$ in $J[i]$ is ————

   (a) $2-i$ (b) 1

   (c) $1+2i$ (d) none of the above

4. The number of units in the ring of complex numbers is ————

   (a) 0 (b) 2

   (c) 1 (d) 4

5. Which of the following is the unique factorization domain?

   (a) $Z[i]$ (b) $Z(\sqrt{-5})$

   (c) (a) and (b) (d) none of the above

6. The content of the polynomial $3x^6+9x-12$ is ————

   (a) 0 (b) 1

   (c) 3 (d) none of the above

7. Let $F[[x]]$ be the ring of formal power series over a field F. Then rad $F[[x]] =$ ————

   (a) 0 (b) 1

   (c) x (d) none of the above

8. Let R be a commutative regular ring. Then the J – radical of a ring R is

(a) {0} (b) {1}

(c) R (d) none of the above

9. A ring R is isomorphic to a subdirect sum of ―――――― if and only if R is without a prime ideal.

(a) ideals (b) integral domain

(c) prime ideals (d) none of the above

10. If $R^\wedge \neq \{0\}$ then the annihilator of the set of zero divisors of R is ――――――

(a) R (b) {0}

(c) $R^\wedge$ (d) none of the above

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If {0} and R are the only two ideals of the commutative ring R with unit element, then prove that R is a field.

Or

(b) If U is an ideal of the ring R, then prove that R/U is a ring and is a homomorphic image of R.

12. (a) Let R be a Euclidean ring and $a, b \in R$, If $b \neq 0$ is not a unit in R, then prove that $d(a) < d(ab)$.

Or

(b) Let p be a prime integer and suppose that for some integer c which is relatively prime to p we can find integers x and y such that $x^2 + y^2 = cp$. Then prove that there exists integers a and b such that $p = a^2 + b^2$.

13. (a) State and prove the division algorithm.

Or

(b) Define primitive polynomial and prove that product of two primitive polynomials is a primitive polynomial.

14. (a) Let I be an ideal of R. Then prove that $I \subseteq rad\,R$ if and only if each element of the coset $1 + I$ has an inverse in R.

Or

(b) For any ring R, prove that the quotient ring $R / Rad\,R$ is without prime radical.

15. (a) An element $a \in R$ is quasi-regular if and only if $a \in I_a$, prove .

Or

(b) Prove that if R is a ring R, $R/radR$ is isomorphic to a subdirect sum of fields.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that every integral domain can be imbedded in a field.

Or

(b) Let R and R' be rings and $\phi : R \to R'$ is a homomorphism of R onto R' with kernel U. Then prove that R' is isomorphic to R/U. Also prove that there is a one-to-one correspondence between the set of ideals of R' and the set of ideals of R which contain U and this correspondence can be achieved by associated with an ideal W' in R' the ideal W in R defined by $W = \{x \in R \mid \phi(x) \in W'\}$. With W so defined, R/W is isomorphic to R'/W'. Prove.

17. (a) Define Euclidean ring and prove that $J[i]$ is an Euclidean ring.

Or

(b) The ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R if and only if a_0 is a prime element of R.

18. (a) State and prove the Eisenstein criterion.

Or

(b) If R is a unique factorization domain and if $p(x)$ is a primitive polynomial in $R[x]$, then prove that it can be factored in a unique way as the product of irreducible elements in $R[x]$.

19. (a) Let I be an ideal of the ring R. Further, assume that the subset $S \subseteq R$ is closed under multiplication and disjoint from I. Then prove that there exists an ideal P which is maximal in the set of ideals which contain I and do not meet S; any such ideal is necessarily prime.

Or

(b) If I is an ideal of the ring R, then prove:

(i) $rad(R/I) \supseteq \dfrac{radR + I}{I}$ and

(ii) Whenever $I \subseteq radR$, $rad(R/I) = (radR)/I$

20. (a) Let $I_1, I_2, \ldots I_n$ be a finite set of ideals of the ring R. If $I_i + I_j = R$ whenever $i \neq j$, then prove that $R / \bigcap I_i \simeq \Sigma \oplus \left(\frac{R}{I_i} \right)$.

Or

(b) If R is a ring for which $R^{\upsilon} \neq \{0\}$, then

(i) $ann\, R^{\upsilon}$ is a maximal ideal of R

(ii) $ann\, R^{\upsilon}$ consists of all zero divisors of R, plus zero

(iii) Whenever R is without prime radical, R forms a field
