

(6 pages)

Reg. No. :

Code No. : 6063

Sub. Code : C 33 M

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2012.

Third Semester

Mathematics

MEASURE AND INTEGRATION

(For those who joined in July 2008 and afterwards)

Time : Three hours

Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

1. Show that the characteristics function of a set is measurable if and only if the set itself is measurable.
2. Show that if $m^*(E) = 0$ then E is measurable.
3. Give an example of a function which is not Riemann integrable.
4. If f is integrable and $a \geq 0$ show that $\int af dx = a \int f dx$.

5. Give an example of a function of bounded variation.
6. Show that $D^-(-f) = -D_-(f)$.
7. Define the integration of a measurable simple function with respect to a measure.
8. Explain how to construct a new measure from a given measure.
9. Show that every countable set has measure zero.
10. If $m^*(A) = 0$ show that $m^*(A \cup B) = m^*(B)$ for any set B .

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If f and g are measurable functions on a measurable set E show that $f + g$ is also measurable.

Or

- (b) For $A \subseteq \mathbb{R}$, let $-A = \{x / -x \in A\}$. Show that A is measurable if and only if $-A$ is measurable.

12. (a) If f is integrable over E , show that $|f|$ is also integrable over E and $\left| \int_E f \right| \leq \int_E |f|$.

Or

- (b) Let f be an integrable function. Show that $\lim_{h \rightarrow 0} \int f(x+h) - f(x) dx = 0$.
13. (a) Bring out the relationship between bounded function and a function of bounded variation with examples.

Or

- (b) Let f be a function of bounded variation on $[a, b]$ and let $a < c < b$. Prove that $T_a^b = T_a^c + T_c^b$.
14. (a) State and prove Lebergue's dominated convergence theorem.

Or

- (b) Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \rightarrow [0, \infty]$ prove that $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

15. (a) Write the usual meaning prove that E_x is measurable.

Or

- (b) Let f be a continuous function and g be a measurable function. Show that the composite function $f \circ g$ is measurable.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Show that there exists a non-measurable set.

Or

- (b) Show that the outer measure of an interval is its length.

17. (a) State and prove Vitali's lemma.

Or

- (b) If f is finite-valued monotone increasing function defined on the finite interval $[a, b]$, show that f' is measurable and $\int_a^b f' d\mu \leq f(b) - f(a)$.

18. (a) let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| d_x < \infty$. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. and its sum $f(x)$ is integrable.

Or

- (b) Prove

- (i) $\int_A f d_x + \int_B f d_x = \int_{A \cup B} f d_x$ where f is an integrable function and A and B are disjoint measurable sets.
- (ii) If f and g are integrable and $f \leq g$ a.e. then $\int f d_x \leq \int g d_x$.

19. (a) Give an example of measures μ and γ on the same measurable space such that none of the relations $\mu \ll \gamma$, $\gamma \ll \mu$, $\mu \perp \gamma$ holds. Explain.

Or

- (b) State and prove Raen – Nikodym theorem.

20. (a) State and prove cara theolory theorem.

Or

- (b) State and prove Fubini's theorem.