(8 Pages) **Reg. No. : ....**

Code No. : 6316 **Sub. Code : PMAM 31**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2021.

Third Semester

MATHEMATICS — CORE

MEASURE AND INTEGRATION

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answers :

1. Which one of the following is not true

   (a) outer measure is defined for all sets of real numbers

   (b) the outer measure of an interval is its length

   (c) outer measure is countably additive

   (d) outer measure is translation in variant

2. If A is a measurable set of finite outer measure that is contained in B then $m^*(B \sim A) - m^*(B)$ is

(a) $-m^*(A)$ (b) $m^*(A)$

(c) $m^*(A \cup B)$ (d) zero

3. For a function f defined on E, $f^-(x)$ is defined by

(a) $\max\{f(x), 0\}$ (b) $\max\{f(x), -f(x)\}$

(c) $\max\{-f(x), 0\}$ (d) $-\max\{f(x), 0\}$

4. Let A and B be any sets, then $\chi_{A \cup B}$ is

(a) $\chi_A + \chi_B - \chi_A \cdot \chi_B$

(b) $\chi_A + \chi_B$

(c) $\chi_A + \chi_B + \chi_{A \cap B}$

(d) $\chi_A \cdot \chi_B$

5. The set E of rational number in [0, 1] is a measurable set of measure

(a) 1 (b) 0

(c) ∞ (d) $\sqrt{2}$

6. Let f be a bounded measurable function on a set of finite measure E, suppose A, B are disjoint measurable subsets of E, then $\int_{AuB} f$ is

(a) $\int_A f$ (b) $\int_B f$

(c) $\int_A f + \int_B f$ (d) $\int_A f - \int_B f$

7. The average value function $Av_h f$ of $[a,b]$ is defined by

(a) $Av_h f(x) = \frac{1}{h} \int_x^{x+h} f \ \forall x \in [a,b]$

(b) $Av_h f(x) = \frac{\left(\int_x^{x+h} f \right)}{x} \ \forall x \in [a,b]$

(c) $Av_h f(x) = \frac{\left(\int_x^{x+h} f \right)}{l-a} \ \forall x \in [a,b]$

(d) $Av_h f(x) = \frac{f(x+h) - f(x)}{h} \ \forall x \in [a,b]$

8. If the function f is monotone on the open interval (a, b) then it is differentiable *a.e.* on (9, 4) this result is known as

(a) Jordan's theorem

(b) Lebesgue's theorem

(c) Mean value theorem

(d) Vital's theorem

9. Which one of the following is not true

(a) Absolutely continuous functions are continuous

(b) Sum of two absolutely continuous functions is absolutely continuous

(c) Composition of absolutely continuous functions is absolutely continuous

(d) Linear combination of absolutely continuous functions is absolutely continuous

10. The function f defined on [0, 1] by $f(x) = \sqrt{x}$ for $10 \le x \le 1$ is

(a) both absolutely continuous and Lipschitz

(b) neither absolutely continuous nor Lispchitz

(c) absolute continuous but not lipchitz

(d) lipchitz but not absolutely continuous

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 250 words.

11. (a) Prove that outer measure is countably subadditive.

Or

(b) State and prove the Borel – Cantelli lemma.

12. (a) Prove that f is measurable if and only if for each open set 0, the inverse image of 0 under f, $f^{-1}(0)$, is measurable.

Or

(b) Let $\{f_n\}$ be a sequence of measurable functions on E that converges point wise a.e. on E to the function f. Prove that f is measurable.

13. (a) Let f be a bounded measurable function on a set of finite measure E. Prove that f is integrable over E

Or

(b) Let f be a non negative measurable function on E. Prove that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

14. (a) Let f be integrable over E and $\{E_n\}_{n=1}^{\alpha}$ a disjoint countable collection of measurable subsets of E whose union is E. Prove that $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$.

Or

(b) Let C be a countable subset of the open interval (a,b). Prove that there is an increasing function on (a,b) that is continuous only at points in $(a,b) \sim C$.

15. (a) Let f be integrable over $[a,b]$. Prove that $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$ for almost all $x \in (a,b)$.

Or

(b) Let μ be a measure. Explain the method of obtaining the outer measure induced by μ.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)
Each answer should not exceed 600 words.

16. (a) Prove that the outer measure of an interval is its length.

Or

(b) State the two continuity preparation of the Lebesgue measure and prove them.

17. (a) Let f and g be measurable functions on E that are finite a.e. on E. Prove that $\alpha f + \beta g$ and fg are measurable on E for any α and β.

Or

(b) State and prove Egoroff's theorem.

18. (a) Let f and g bounded measurable functions on a set of finite measure E. Prove that $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$ for any α and β and show that if $f \leq g$ on E then $\int_E f \leq \int_E g$.

Or

(b) State and prove the boundary convergence theorem.

19. (a) State and prove the Lebsegue dominated convergence theorem.

Or

(b) If the function f is monotone on the open interval (a,b), prove that it is differentiable almost everywhere on (a,b).

20. (a) Let the function f be absolutely continuous on $[a,b]$. Prove that f is the difference of increasing absolutely continuous functions and is of bounded variation.

Or

(b) State Hahn's lemma without proof and deduce the Hahn Decomposition theorem with proof.