(6 pages) **Reg. No. :** ....

Code No. : 9368 **Sub. Code : HMAM 42**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2015.

Fourth Semester

Mathematics

MEASURE AND INTEGRATION

(For those who joined in July 2012 onwards)

Time : Three hours Maximum : 75 marks

SECTION A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. For any set A and any number y, $m^*(A+y)=$ ____________

(a) $m^*(A)+y$ (b) $m^*(A)-y$

(c) $m^*(A)$ (d) $m^*(\{y\})$.

2. The translate of a measurable set is ____________

(a) translate

(b) measurable

(c) integrable

(d) not necessarily measurable.

3. Let φ and ψ be simple functions defined on a set of finite measure. If $\varphi \leq \psi$ on E, then ____________

(a) $\int_E \varphi \leq \int_E \psi$ (b) $\int_E \varphi \leq \int_E \psi$ a.e.

(c) $\int_E \varphi = \int_E \psi$ (d) $\int_E \varphi = \int_E \psi$ a.e.

4. Let f be a bounded measurable function on a set of finite measure E. Then ____________

(a) $\left|\int_E f\right| \leq \int_E |f|$ (b) $\left|\int_E f\right| = \int_E |f|$

(c) $\left|\int_E f\right| \geq \int_E |f|$ (d) $\left|\int_E f\right| > \int_E |f|$

5. Let f be an increasing function on $[a,b]$. Then $m^*\{x \in (a,b) \mid \overline{D}f(x)=\infty\}=$

(a) ∞ (b) (a,b)

(c) 0 (d) $b-a$

6. Let f be Lipschitz function on $[a,b]$ and $|f(u)-f(v)| \leq c|u-v|$ for all u,v in $[a,b]$. Then $T \vee (f) \leq$ ____________

(a) $b-a$ (b) $c(b-a)$

(c) $c(b+a)$ (d) $a+b+c$

7. Let (X, M, μ) be a measurable space. If A and B are measurable sets, $A \subseteq B$ and $\mu(A)=0$, then $\mu(B \sim A)=$ ______________

(a) $\mu(A)$ (b) $\mu(B)$

(c) $\mu(B)+\mu(B^c)$ (d) $\mu(X)$.

8. If M contains all subsets of sets of measure zero, then the measurable space (X, M, μ) is called ______________

(a) complete (b) zero measurable

(c) finite measurable (d) incomplete.

9. Let (X, M, μ) be a measurable space and ψ, a non negative simple function on X. If A and B are disjoint measurable subsets of X, then $\int_{A \cup B} \psi \, d\mu$ ______________

(a) $\int_A \psi \, d\mu$ (b) $\int_B \psi \, d\mu$

(c) $\int_A \psi \, d\mu + \int_B \psi \, d\mu$ (d) $\int_A \psi \, d\mu - \int_B \psi \, d\mu$.

10. Let (X, M, μ) be a measurable space and $\{h_n\}$ a sequence of non negative integrable functions on X. Suppose that $\{h_n(x)\} \to 0$ for almost all x in X. Then $\{h_n\}$ is uniformly integrable and tight if and only if $\lim_{n \to \infty} \int_X h_n \, d\mu =$ ______________

(a) ∞ (b) 1

(c) n (d) 0

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that any countable set has outer measure zero.

Or

(b) Show that Lebesgue measure is countably additive.

12. (a) State and prove Egoroff's theorem.

Or

(b) Let f be a bounded measurable function on a set of finite measure E. Show that f is integrable over E.

13. (a) Let f be an increasing function on $[a,b]$. Prove that f' is integrable over $[a,b]$ and $\int_a^b f' \leq f(b)-f(a)$.

Or

(b) Let the function f be of bounded variation on the closed, bounded interval $[a,b]$. Prove that f is the difference of two increasing function on $[a,b]$.

14. (a) Prove that the union of countable collection of measurable sets is measurable.

Or

(b) State and prove Borel-Cantelli lemma.

15. (a) Show that if f is a non negative measurable function on X, then

$\int_E f\,d\mu = 0$ if and only if $f = 0$ a.e. on X.

Or

(b) Show that if f is integrable over X, then f is integrable over every measurable subset of X.

SECTION C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If a σ-algebra of subsets of R contains intervals of the form (a,∞) prove that it contains all intervals.

Or

(b) Show that a set E is measurable if and only if for each $\in > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O \sim F) < \in$.

17. (a) State and prove simple-approximation theorem.

Or

(b) State and prove Lusin's theorem.

18. (a) State and prove Lebesgue's theorem.

Or

(b) Let the function f be continuous on the closed bounded interval $[a,b]$. Prove that f is absolutely continuous on $[a,b]$ if and only if the family of divided difference functions $\{Diff_h t\}_{0<h\leq 1}$ is uniformly integrable over $[a,b]$.

19. (a) State and prove Hahn's lemma.

Or

(b) Let γ be a signed measure on the measurable space (X,M). Then show that there is a positive set A for γ and a negative set B for γ for which $X = A \cup B$ and $A \cap B = \phi$.

20. (a) State and prove the Vitali convergence theorem.

Or

(b) State and prove Fatou's lemma on a general measure space.
