(8 pages) **Reg. No. :** ....

Code No. : 7846 **Sub. Code : PMAM 33**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2019.

Third Semester

Mathematics — Core

ADVANCED ALGEBRA — I

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If V is finite dimensional and W is a subspace of V, then $\dim A(W) =$ ————

   (a) $\dim V - \dim W$

   (b) $\dim W - \dim V$

   (c) $\dim \hat{V} - \dim \hat{W}$

   (d) None

2. If $u, v, \in V$ then u is said to be orthogonal to v if ————

   (a) $(u,v) = 0$ (b) $(u,v) \neq 0$

   (c) $(u,v) = 1$ (d) $(u, v) \neq 1$

3. If $T \in A(V)$ and if $S \in A(V)$ is regular then ————

   (a) $r(T) < r(STS^{-1})$

   (b) $r(T) > r(STS^{-1})$

   (c) $r(T) = r(STS^{-1})$

   (d) $r(STS^{-1}) > r(S)$

4. If V is finite dimensional over F, and if $T \in A(V)$ is singular then there exists an $S \neq 0$ in $A(V)$ such that ————

   (a) $ST = TS = I$ (b) $ST = TS = 0$

   (c) $ST \neq TS$ (d) $ST = TS = \{0\}$

5. A subspace W of V is invariant under $T \in A(V)$ if ————

   (a) $WT \supset W$ (b) $WT = W$

   (c) $WT \subset W$ (d) $WT \neq W$

6. If $T \in A(V)$ is nilpotent, then ———— is called the index of nipotence of T if $T^k = 0$ but $T^{k-1} \neq 0$

(a) $K-1$ (b) K

(c) $K+1$ (d) $K-2$

7. If A is invertible then $tr(A \subset A^{-1}) =$

(a) $tr\,A$ (b) $tr\,A^{-1}$

(c) $tr\,AA^{-1}$ (d) $tr\,C$

8. For $A, B \in F_n$, $\det(AB) =$ ————

(a) $\det A + \det B$ (b) $\det A - \det B$

(c) $(\det A)(\det B)$ (d) none

9. If $T \in A(V)$ then the Hermitian adjoint of T, T^* is defined by $(uT, v) =$ ———— for all $u, v \in V$

(a) (uT^*, v) (b) (u, vT^*)

(c) (u, Tv) (d) (v, T_u^*)

10. $T \in A(V)$ is unitary if and only if ————

(a) $TT^* = 0$ (b) $TT^* \neq 0$

(c) $TT^* = T^*T$ (d) $TT^* = 1$

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If V is finite dimensional, then prove that ψ is an isomorphism of V onto $\hat{V}$.

Or

(b) If $u, v \in V$, then prove that $|(u, v)| \leq \|u\|\|v\|$.

12. (a) Show that if V is finite dimensional over F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not zero.

Or

(b) Prove that the element $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if for some $v \neq 0$ in V, $vT = \lambda v$.

13. (a) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F then show that T satisfies a polynomial of degree n over F.

Or

(b) Let $T \in A(V)$ have all its distinct characteristic roots $\lambda_1, \lambda_2, ..., \lambda_k$ in F, then a basis of V can be found in which the matrix T is of the form $\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$ where each $J_i = \begin{pmatrix} Bi_1 & & & \\ & Bi_2 & & \\ & & \ddots & \\ & & & Bi_{r_i} \end{pmatrix}$ and where $Bi_1, Bi_2, ... Bi_{r_i}$ are basic Jordan blocks belonging to λ_i.

14. (a) For $A, B \in F_n$ and $\lambda \in F$, prove the following :

(i) $tr(\lambda A) = \lambda tr\, A$

(ii) $tr(A+B) = trA + trB$

(iii) $tr(AB) = tr(BA)$.

Or

(b) Prove that every $A \in F_n$ satisfies its secular equation.

15. (a) Prove that if $\{v_1, v_2, ..., v_n\}$ is an orthonormal basis of V and if the matrix of $T \in A(V)$ in this basis is (α_{ij}) then the matrix of T^* in this basis is (β_{ij}) where $\beta_{ij} = \overline{\alpha_{ji}}$.

Or

(b) Show that if N is normal and $AN = NA$ then $AN^* = N^* A$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

16. (a) Let V and W be two vector spaces over F of dimensions m and n respectively. Prove that $Hom(V, W)$ is a vector space over F. Also find the dimension of $Hom(V, W)$ over F.

Or

(b) Let V be a finite dimensional inner product space then prove that V has an orthogonal set as a basis.

17. (a) Prove that if V is finite dimensional over F, then $T \in A(V)$ is regular iff T maps V onto V.

Or

(b) Show that if V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1, v_2, \ldots v_n$ and the matrix $m_2(T)$ in the basis $w_1, w_2, \ldots w_n$ of V over F, then there is an element $C \in F_n$ such that $m_2(T) = Cm_1(T)C^{-1}$. Infact, if S is the linear transformation of V defined by $v_i s = w_i$ for $i = 1, 2, \ldots n$ then C can be chosen to be $m_1(s)$.

18. (a) If $T \in A(V)$ has all its characteristics roots in F, then prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Show that two nilpotent linear transformations are similar if and only if they have the same invariants.

19 (a) State and prove Cramer's rule.

Or

(b) Show that A is invertible if and only if $\det A \neq 0$.

20. (a) If $T \in A(V)$ then prove that $T^* \in A(V)$. More over

(i) $(T^*)^* = T$

(ii) $(S+T)^* = S^* + T^*$

(iii) $(\lambda S)^* = \bar{\lambda} S^*$

(iv) $(ST)^* = T^* S^*$.

Or

(b) Prove that if N is a normal linear transformation on V, then there exists an orthonormal basis consisting of characteristic vectors of N, in which the matrix of N is diagonal, Equivalently, if N is a normal matrix there exists a unitary matrix U such that $UNU^{-1}(= UNU^*)$ is diagonal.