(7 pages) **Reg. No. :** ....

Code No. : 10421 E **Sub. Code : CMMA 41**

B.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2024.

Fourth Semester

Mathematics – Core

ABSTRACT ALGEBRA

(For those who joined in July 2021 – 2022)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. In $(\mathbb{Z}_7 - \{0\}, \odot)$, the inverse of 3 is ————.

   (a) 2 (b) 3

   (c) 4 (d) 5

2. Which of the following are not subgroups of $(C^*, \cdot)$?

   (a) $\{1, i, -1, -i\}$ (b) $\{1, -1\}$

   (c) $\{1\}$ (d) $\{i, -i\}$

3. If $H = \{0, 4, 8\}$ in the group $(\mathbb{Z}_{12}, \oplus)$, then $H + 4 =$ ————.

   (a) $\mathbb{Z}_{12}$ (b) φ

   (c) H (d) $H + 1$

4. The set of generators of the group $(\mathbb{Z}_6, \oplus)$ is ————.

   (a) $\{1, 5\}$ (b) $\{1, 2, 4\}$

   (c) $\{1, 2, 5\}$ (d) $\{2, 3, 5\}$

5. A mapping $\varphi : (G, \cdot) \to (G', \cdot)$ is called a homomorphism if ———— $\forall\, a, b, \in G$.

   (a) $\varphi(a + b) = \varphi(a) + \varphi(b)$

   (b) $\varphi(ab) = \varphi(a)\varphi(b)$

   (c) $\varphi(a - b) = \varphi(a) - \varphi(b)$

   (d) none of these

6. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} =$ ————

   (a) $(1\ 2\ 3)(4\ 5)$

   (b) $(1\ 2)(3\ 4\ 5)$

   (c) $(1\ 4)(3\ 5)$

   (d) $(1\ 2\ 3\ 4\ 5)$

7. $\mathbb{Z}_{12}$ is not an integral domain if ————.

(a) it has no zero divisors

(b) 12 is not prime

(c) 4 is a zero divisor

(d) (b) and (c)

8. $f(x), g(x) \in \mathbb{Z}_4[x]$ is defined by $f(x) = x^2 + 2x + 3$ and $g(x) = 3x^2 + 2x + 2$. Then degree of $(f(x) + g(x))$ is ————.

(a) 0 (b) 2

(c) 4 (d) 1

9. The map $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 + 3$ $\forall\, x \in \mathbb{Z}$ is ————.

(a) a ring homomorphism

(b) not a ring homomorphism

(c) a ring isomorphism

(d) a ring epimorphism

10. A homomorphism $\varphi : R \to R'$ (R, R' are rings) is an isomorphism if it is ————.

(a) onto

(b) 1 – 1 and onto

(c) 1 – 1

(d) none of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let G be a group and $a, b \in G$. Prove that

(i) $O(a) = O(a^{-1})$

(ii) $O(a) = O(b^{-1}ab)$

(iii) $O(ab) = O(ba)$.

Or

(b) If H and K are subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

12. (a) Prove that a subgroup of a cyclic group is cyclic.

Or

(b) State and prove Euler's theorem.

13. (a) Prove that any permutation can be expressed as a product of disjoint cycles.

Or

(b) Let G and G' be groups and $f : G \to G'$ be a homomorphism. Prove that f is $1-1$ iff $\ker f = \{e\}$.

14. (a) Prove that the characteristics of an integral domain is either 0 or a prime number.

Or

(b) In $M_2 - (R)$, show that the set $S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \Big/ a, b \in R \right\}$ is only a left ideal but not a right ideal.

15. (a) Prove that any homomorphism of a field to itself is either $1-1$ or maps every element to zero.

Or

(b) Show that the map $f : \mathbb{Z} \to \mathbb{Z}_n$ defined by $f(x) = r \ \forall \ x \in \mathbb{Z}$, where $x = qn + r, 0 \le r < n$ is a homomorphism.

PART C — $(5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let A and B be two subgroups of a group G. Prove that AB is a subgroup of G if and only if $AB = BA$.

Or

(b) Prove that the union of two subgroups of a group G is a subgroup if and only if one is contained in the other.

17. (a) If H and K are two finite subgroups of a group G, prove that $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.

Or

(b) Let H be a subgroup of a group G. Prove that the number of left cosets of H is the same as the number of right cosets of H in G.

18. (a) For any group G, show that $Aut\, G$ is a group and $I(G)$ is a normal subgroup of $Aut\, G$.

Or

(b) State and prove Cayley's theorem.

19. (a) Let R be a commutative ring with identity. Prove that an ideal M of R is maximal if and only if $\dfrac{R}{M}$ is a field.

Or

(b) Let R be a ring and I be a subgroup of $(R, +)$. The multiplication in $\dfrac{R}{I}$ given by a subgroup of $(R, +)$. The multiplication is $\dfrac{R}{I}$ given by $(I + a)(I + b) = I + ab$ is well defined iff I is an ideal of R.

20. (a) State and prove the fundamental theorem of homomorphism of rings.

Or

(b) Let R, R' and R'' be rings. If $f : R \to R'$ and $g : R' \to R''$ are homomorphisms, prove that $g \circ f : R \to R''$ is a homomorphism.

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