(8 pages) **Reg. No. :** ....

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M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2019.

Third Semester

Mathematics – Core

MEASURE AND INTEGRATION

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. E is measurable if

(a) If A is any set then $m^*(A) = m^*(A \cap E)$

(b) There exists a G_δ set $G \subset E$ such that $m^*(G \vee E) = 0$

(c) For each $\in > 0$ there exists a closed set $F \subset E$ for which $m^*(E \vee F) = 0$

(d) None of these

2. A countable set has outer measure ————.

(a) 0 (b) 1

(c) ∞ (d) finite

3. $\{x \in E / \mathfrak{C}(x) > c\}$ = ————.

(a) $\cup_{n=1}^{\infty} \{x \in E / \mathfrak{C}(x) \geq c + \frac{1}{n}\}$

(b) $\cap_{n=1}^{\infty} \{x \in E / \mathfrak{C}(x) \geq c + \frac{1}{n}\}$

(c) $\cup_{n=1}^{\infty} \{x \in E / \mathfrak{C}(x) > c + \frac{1}{n}\}$

(d) $\cap_{n=1}^{\infty} \{x \in E / \mathfrak{C}(x) > c + \frac{1}{n}\}$

4. Which one of the following is false?

(a) f is measurable if $\mathfrak{C}^{-1}(0)$ is measurable for any open set O or R

(b) A continuous real valued function on its measurable domain is measurable.

(c) A monotonic function defined on an interval is measurable.

(d) Composition of any two measurable functions is always measurable.

5. Let f be a bounded measurable function on E. Let then $E = \{$Rationals in $[0,1]$ only let $f = 1.x_E$ on $[0,1]$.

(a) $\int\limits_{[0,1]} f = 0$

(b) $\int\limits_{[0,1]} f = 1$

(c) Integral does not exist on [0, 1]

(d) None of these

6. Let the non-negative function f be integrable over E. Then $\mathfrak{g}$ is ———— on E.

(a) finite $a - e$ (b) finite

(c) Zero (d) Constant

7. Let $\mathfrak{g}$ be monotonic function on (a, b). Then $\mathfrak{g}$ is continuous except possibly at

(a) Countable number of points in *(a, b)*

(b) Finite number of points in *(a, b)*

(c) Uncountable number of points in *(a, b)*

(d) None of the above

8. A closed interval *[c,d]* is said to be non-degenerate is

(a) $c > d$ (b) $c < d$

(c) $c = d$ (d) None

9. $\mathfrak{F}$ lip, $\mathfrak{F}$AC, $\mathfrak{F}$BV, denote the family of functions on [a, b] that are Lipschitz, absolutely continuous and bounded variation respectively. Then

(a) $\mathfrak{F}$ lip $\subseteq$ $\mathfrak{F}$AC $\subseteq$ $\mathfrak{F}$BV

(b) $\mathfrak{F}$ lip $\subseteq$ $\mathfrak{F}$BV $\subseteq$ $\mathfrak{F}$AC

(c) $\mathfrak{F}$AC $\subseteq$ $\mathfrak{F}$BV $\subseteq$ $\mathfrak{F}$ lip

(d) $\mathfrak{F}$BV $\subseteq$ $\mathfrak{F}\cap$C $\subseteq$ $\mathfrak{F}$ lip

10. The counting measure on an uncounatable set is ————

(a) σ-finite (b) not σ-finite

(c) σ-infinite (d) finite

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Show that for any bounded set E, there exists a G_δ set G for which $E \subset G$ and $m^*(E) = m^*(G)$.

Or

(b) Prove that the translate of a measurable set in measurable.

12. (a) Prove that a monotone function defined on an interval is measurable.

Or

(b) Let $\{fn\}$ be a sequence of measurable functions on E that converges pointwise almost every where on E to the functions $\mathfrak{S}$, Then show that f is measurable.

13. (a) Let E have measure zero. Let $\mathfrak{S}$ be a bounded function on E. Then show that $\mathfrak{S}$ is measurable and $\int_E f = 0$.

Or

(b) Let $\{fn\}$ be a sequence of bounded measurable functions on a set of finite measure E. If $\{fn\}$ converges to $\mathfrak{S}$ uniformly on E, then show that $\lim_{n\to\infty} \int_E fn = \int_E f$

14. (a) Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then show that $\int_{A\cup B} f = \int_A f + \int_B f$

Or

(b) Let f be an increasing function on the closed, bounded interval $[a, b]$. Then show that for each $\alpha > 0, m^*\{x \in (a, b) / \overline{D} f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$ and $m^*\{x \in (a, b) / \overline{D} f(x) = \infty\} = 0$.

15. (a) Let the function f be absolutely continuous on the closed, bounded interval $[a / b]$. Then show that f is the difference of increasing absolute continuous functions and, in particular, f is of bounded variation.

Or

(b) Let γ be a signed measure on the measurable space $(X_1 \mathcal{M})$. Then show that every measurable subset of a positive set is itself a positive set and the union of countable collection of positive sets is positive.

PART C — $(5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that outer measure of intervals is its length.

Or

(b) Prove that the collection of lebesgue measurable sets form a σ-algebra.

17. (a) State and prove Lusin's theorem.

Or

(b) (i) If f is an entended real values measurable function on E and $f = g$ $a-e$ on E, then show that g is measurable on E.

(ii) If f and g are measurable functions on E that are finite a-e on E then show that $\alpha f + \beta g$ is measurable on E for any α and β and also show that fg is measurable on E.

18. (a) Let f and g be bounded measurable functions on a set of finite measure E. Then show that for any α and β, $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$. More over, if $f \le g$ on E, show that $\int_E f \le \int_E g$.

Or

(b) State and prove Bounded Convergence theorem.

19. (a) State and prove Vitali Covering Lemma.

Or

(b) (i) For $a \le u < v \le b$, show that $\int_u^v \text{Diff}_h\, f(x)\,dx = AV_h f(v) - AV_h f(u)$.

(ii) Let f be an increasing function on closed bounded interval $[a, b]$. Then show that f' is integrable on $[a, b]$ and $\int_a^b f' \le f(b) - f(a)$.

20. (a) State and prove Hahn's Lemma and the Hahn decomposition theorem.

Or

(b) Prove the following:

(i) Let $\mathcal{S}$ be a collection of subsets of set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\phi) = 0$ and for $E \in \mathcal{S}$, $E \ne \phi$, define $\mu^*(E) = \inf \sum_{k=1}^{\infty} (\mu(Eu)$ where the infimum is taken over all countable collections $\{Eu\}_{n=1}^{\infty}$ of sets is $\mathcal{S}$ that cover E. The show that the set fucntion $\mu^* : 2^X \to [0, \infty]$is an outer measure called the outer measure induced by μ.

(ii) Let $\mu : \mathcal{S} \to [0, \infty]$ be a set function defined on a collections of $\mathcal{S}$ of subsets of a set X and $\bar{\mu} : \mathcal{M} \to [0, \infty]$ the cartheodary measure induced by μ. Let $E \subset X$ for which $\mu^*(E) < \infty$. Then show that there exists a subset A of X for which $A \in \mathcal{S}_{\sigma\delta}$, $E \subseteq A$ and $\mu^*(E) = \mu^*(A)$. Furthermore, if E and each set in $\mathcal{S}$ is measurable with respect to μ^*, then so is A and $\bar{\mu}(A \sim E) = 0$.
