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M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2023.

First Semester

Mathematics – Elective – II

ANALYTIC NUMBER THEORY

(For those who joined in July 2023 onwards)

Time : Three hours

Maximum : 75 marks

PART A — ($15 \times 1 = 15$ marks)

Answer ALL questions.

Choose the correct answer :

1. $d/n \Rightarrow$ _____.
(a) n/d (b) $d+1/n$
(c) ad/an (d) $d/n+1$
2. $(a, b) =$ _____.
(a) (b, a) (b) $(a+1, b)$
(c) $(a, b+1)$ (d) $(a+1, b+1)$

3. $(8, 20) =$ _____.
(a) 2 (b) 8
(c) 20 (d) 4
4. $\mu(4) =$ _____.
(a) 1 (b) 0
(c) 8 (d) -1
5. $I(7) =$ _____.
(a) 0 (b) 1
(c) 4 (d) -1
6. $\phi(6) =$ _____.
(a) 1 (b) -3
(c) 2 (d) 4
7. If f is multiplicative, then $f(1) =$ _____.
(a) 0 (b) -1
(c) 1 (d) 6
8. $\lambda(1) =$ _____.
(a) 6 (b) 3
(c) -1 (d) 1



9. $d(n)$ is the _____.

- (a) number of divisors of n
- (b) n
- (c) n^2
- (d) sum of the divisors of n

10. Euler's constant $c =$ _____.

- (a) $\lim_{n \rightarrow \infty} \log n$
- (b) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right)$
- (c) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right)$
- (d) n^2

11. If $f(x) = O(g(x))$, $g(x) > 0$, then _____.

- (a) $\frac{f(x)}{g(x)}$ is bounded for all $x \geq a$
- (b) $\frac{g(x)}{f(x)}$ is bounded for all $x \geq a$
- (c) $\frac{f(x)+1}{g(x)+1}$ is bounded for all $x \geq a$
- (d) $f(x)g(x)$ is bounded for all $x \geq a$

12. If $f(x) \sim g(x)$ as $x \rightarrow \infty$, then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} =$ _____.

- (a) 0
- (b) 7
- (c) 10
- (d) 1

13. If $\alpha(n)$ is the characteristic function of prime, then $\alpha(7) =$ _____.

- (a) 1
- (b) 0
- (c) -1
- (d) 2

14. $U\left(\frac{1}{2}\right) =$ _____.

- (a) -1
- (b) -2
- (c) 0
- (d) 1

15. For $x > 0$, the chebyshev's χ -function $\chi(x) =$ _____.

- (a) $\wedge(n)$
- (b) $\sum_{n \leq x} \wedge(n)$
- (c) $\sum_{n=0}^{\infty} \wedge(n)$
- (d) $\prod_{n \leq x} \wedge(n)$



PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that every integer $n > 1$ is either a prime number or a product of prime numbers.

Or

- (b) If $(a, b) = 1$, then prove that $(a+b, a^2 - ab + b^2)$ is either 1 or 3.

17. (a) If $n \geq 1$, prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Or

- (b) If $n \geq 1$ prove that $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

18. (a) If f and g are multiplicative, then prove that their Dirichlet product $f * g$ is also multiplicative.

Or

- (b) Let $f(n) = [\sqrt{n}] - [\sqrt{n-1}]$. Then prove that f is multiplicative but not completely multiplicative.

19. (a) For all $x \geq 1$, prove that $\sum_{n \leq x} \sigma_1(n) = \frac{1}{2} (2)x^2 + O(x \log x)$.

Or

- (b) If $\beta > 0$, let $\delta = \min\{0, 1 - \beta\}$. If $x > 1$, prove that $\sum_{n \leq x} \sigma_{-\beta}(n) = \xi(\beta+1) + O(x^\delta)$ if $\beta \neq 1$ = $\xi(2)x + O(\log x)$ if $\beta = 1$.

20. (a) For $x \geq 1$, prove that $\sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right] = 1$ and

$$\sum_{n \leq x} \wedge(n) \left[\frac{x}{n} \right] = \log[x]!$$

Or

- (b) If $x \geq 2$, prove that $\log[x]! = x \log x - x + O(\log x)$ and hence $\sum_{n \leq x} \wedge(n) \left[\frac{x}{n} \right] = x \log x - x + O(\log x)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) State and prove Fundamental theorem of arithmetic.

Or

- (b) State and prove the Euclidean algorithm.



22. (a) For $n \geq 1$, prove that $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

Or

- (b) If f is an arithmetical function with $f(1) \neq 0$, then prove that there is a unique arithmetical function f^{-1} such that $f * f^{-1} = f^{-1} * f = I$.

Moreover f^{-1} is given by $f^{-1}(1) = \frac{1}{f(1)}$ and

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d < n}} f\left(\frac{n}{d}\right) f^{-1}(d), \text{ for } n > 1.$$

23. (a) Let f be multiplicative. Then prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n)f(n)$ for all $n \geq 1$.

Or

- (b) Prove that $\sigma_1(n) = \sum_{d|n} \phi(d) o\left(\frac{n}{d}\right)$ and derive a generalization involving $\sigma_a(n)$.

24. (a) State and prove Euler's summation formula.

Or

- (b) For all $x \geq 1$, prove that $\sum_{n \leq x} d(n) = x \log x + (2c-1)x + o(\sqrt{x})$.

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25. (a) For $x > 0$, prove that

$$0 \leq \frac{\chi(x)}{x} - \frac{\gamma(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x} \log 2}.$$

Or

- (b) Prove that the following are logically equivalent

(i) $\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1$

(ii) $\lim_{x \rightarrow \infty} \frac{\gamma(x)}{x} = 1$

(iii) $\lim_{x \rightarrow \infty} \frac{\chi(x)}{x} = 1.$

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