(8 Pages) **Reg. No. : ....**

Code No. : 5330 **Sub. Code : PMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2022

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If the complex number C are normed linear space then

(a) $\|x\| = \|x\|$ (b) $\|x\| = |x|$

(c) $\|x\| \leq |x|$ (d) $\|x\| \geq |x|$

2. A normed linear space has one of the following property

(a) $\|\alpha x\| = |\alpha| \|x\|$ (b) $\|\alpha x\| = \alpha x$

(c) $\|\alpha x\| \leq |\alpha| \|x\|$ (d) $\|\alpha x\| \geq |\alpha| \|x\|$

3. The conjugate space of a normed linear space is

(a) linear space

(b) normed linear space

(c) banach space

(d) none of these

4. A banach space B is reflexive iff

(a) B^* is not relfexive

(b) B^* is symmetric

(c) B^* is relfexive

(d) B^* is transitive

5. A mapping $T \to T^*$ then $(\alpha T_1 + \beta T_2)^*$ is

(a) I (b) $\alpha T_1{}^* + \beta T_2{}^*$

(c) $\alpha T_1 + \beta T_2$ (d) $\alpha T_2{}^* + \beta T_1{}^*$

6. If x and y are orthogonal in an inner product space, then

(a) $(x, y) = \|x\| \|y\|$ (b) $(x, y) = 0$

(c) $(x, y) = 1$ (d) $(x, y) \leq \|x\|$

7. An orthonormal set in a Hilbert space H consists of

(a) orthogonal vectors

(b) mutually orthogonal unit vectors

(c) orthogonal unit vectors

(d) none of these

8. The Bessel's inequality is

(a) $\Sigma|x_i e_i|^2 \geq \|x\|^2$ (b) $\Sigma|x_i e_i|^2 \leq \|x\|^2$

(c) $\Sigma|x_i e_i|^2 > \|x\|^2$ (d) $\Sigma|x_i e_i|^2 < \|x\|^2$

9. An operator T on H is unitary then

(a) $TT^* = I$ (b) $TT^* = T^*T = I$

(c) $TT^* \neq T^*T$ (d) None of these

10. If $\det([\alpha_{ij}]) \neq 0$ iff

(a) $[\alpha_{ij}]$ is singular

(b) $[\alpha_{ij}]$ is non singular

(c) $[\alpha_{ij}]$ identity matrix

(d) none of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) State and prove Minkowski's unequality.

Or

(b) Prove that if M is a closed linear subsapce of a normed liner space N and x_0 is a vector not in M, then there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

12. (a) State and prove closed graph theorem.

Or

(b) If P is a projection on a Banach space B, and if M and N are its range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$ – prove.

13. (a) If B is a complex banach space whose norm obeys the parallelogram law, and if an inner product is defined on B by $4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ then prove that B is a Hilbert space.

Or

(b) If M is a closed linear subspace of a Hilbert space H then $H = M \oplus M^{\perp}$ – prove.

14. (a) Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then $\sum_{i=1}^{n} |(x_i e_i)|^2 \le \|x\|^2$ further $x - \sum_{i=1}^{n} (x_i e_i) e_i \perp e_j$ for each j. – Prove.

Or

(b) Prove that if A_1 and A_2 are self-adjoint operators on H, then their product $A_1 A_2$ is self-adjoint iff $A_1 A_2 = A_2 A_1$.

15. (a) If P is the projection on a closed linear subspace M of H then M is invariant under an operator $T \Leftrightarrow TP = PTP$ - Prove.

Or

(b) It T is normal, then the M_i's are pairwise orthogonal.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If N and N' are normed linear spaces, then the set $B(N, N')$ of all continuous linear transformations of N and N' is itself a normed linear space with respect to point wise linear operations and the norm defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \le 1\}$ further, if N' is a Banach space, then $B(N, N')$ is also a Banach space – Prove.

Or

(b) Let M be a liner subspace of a normed linear space N and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0, then prove that f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

17. (a) State and prove open mapping theorem.

Or

(b) Prove that if N is a normed linear space, then the closed unit sphere in S^* in N^* is a compact Hausdroff space in the weak topology.

18. (a) State and prove uniform boundedness theorem.

Or

(b) A closed convex subset C of a Hilbert space if contains a unique vector of smallest norm – prove.

19. (a) Let H be a Hilbert space and let f be an arbitrary functional in H^*. Then prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every $x \in H$.

Or

(b) If A is a positive operator on H, then prove that If A is an singular in particular, $I + T^*T$ and $I + TT^*$ are non-singular for an arbitrary operator T on H.

20. (a) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and N_1N_2 are normal.

Or

(b) If P is a projection on H with range M and null space N then $M \perp N \Leftrightarrow P$ is self-adjoint and $N = M^{\perp}$ - Prove.