(7 pages) **Reg. No. :** ....

Code No. : 5383 **Sub. Code : ZMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2023.

Fourth Semester

Mathematics — Core

ADVANCED ALGEBRA — II

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. What is the degree of $\sqrt{2}\sqrt{3}$ over Q?

   (a) 1 (b) 2

   (c) 3 (d) 4

2. If $a, b \in K$ are algebraic over F of degrees 8 and 3 respectively then the degree of $F(a, b)$ is

   (a) 5 (b) 16

   (c) 24 (d) 80

3. If $f(x) \in F[x]$ is irreducible and if characteristics of F is zero then $f(x)$ has

   (a) a unique root (b) more than one root

   (c) a multiple root (d) no multiple roots

4. If E is the splitting field of $f(x) = x^3 - 2$ over the field of rational numbers then $[E : F]$ is

   (a) 3 (b) 6

   (c) 2 (d) 4

5. If G is a group of automorphisms of K then the fixed field of G is

   (a) $\{a \in K / \sigma(a) = a \ \forall \ \sigma \in G\}$

   (b) $\{a \in K / \sigma(a) = 0 \ \forall \ \sigma \in G\}$

   (c) $\{a \in G / \sigma(a) = a \ \forall \ \sigma \in K\}$

   (d) $\{a \in G / \sigma(a) = 0 \ \forall \ \sigma \in K\}$

6. Let K be the field and let F be the subfield of K. Then $G(K, F)$ is

   (a) the set of all automorphisms of K

   (b) the set of all automorphisms of K leaving every element of F fixed

   (c) the set of all automorphisms of F leaving every element of K fixed

   (d) the set of all automorphisms of F

7. The multiplicative group of non zero elements of a finite field is

(a) cyclic

(b) need not be cyclic

(c) a permutation group

(d) a sub ring

8. If $\phi_n(x)$ is the n^{th} cyclotomic polynomial then $\phi_1(x)+\phi_2(x)$ is

(a) 0 (b) $x-1$

(c) $2x$ (d) x^2+x+1

9. The irreducible polynomials over the field of real numbers are of degree less than

(a) 0 (b) 1

(c) 2 (d) 3

10. Let H be the Hurwitz ring of integral quaternions. If $a \in H$ then $a^{-1} \in H$ iff $N(a)=$

(a) ∞ (b) 0

(c) 1 (d) not defined

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define algebraic over F. Also prove that the elements in K which are algebraic over F form a subfield of K.

Or

(b) If $a \in K$ are algebraic over F of degrees m and n respectively and if m and n are relatively prime then prove that $F(a, b)$ is of degree mn over F.

12. (a) Prove that a polynomial of degree n over a field can have at most n roots in any extension field.

Or

(b) Prove that for any $f(x), g(x) \in F[x]$ and any $\alpha \in F$,

(i) $(f(x)+g(x))' = f'(x)+g'(x)$

(ii) $(\alpha f(x))' = \alpha f'(x)$ and

(iii) $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$.

13. (a) Define the group of automorphisms of K relative to F and show that $G(K, F)$ is a subgroup of the group of all automorphisms of K.

Or

(b) If K is a finite extension of F then prove that $o(G(K, F)) \leq [K:F]$.

14. (a) Prove that for every prime number p and every positive integer m there is a unique field having p^m elements.

Or

(b) Let K be a field and Let G be a finite subgroup of the multiplicative group of nonzero elements of K. Then prove that G is a cyclic group.

15. (a) Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Then prove that $D = C$.

Or

(b) State and prove Lagrange Identity.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) State and prove algebraic extension theorem.

Or

(b) Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

17. (a) If $p(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$ and is irreducible over F then prove that there is an extension E of F such that $[E:F] = n$, in which $p(x)$ has a root.

Or

(b) State and prove simple extension theorem.

18. (a) State and prove fundamental theorem of Galois Theory.

Or

(b) Define a normal extension of a field F. Prove that $[K:K_H] = o(H)$ and $H = G(K, K_H)$ where K is a normal extension of F and K_H is the fixed field of H.

19. (a) Let G be a finite abelian group enjoying the property that the relation $x^n = e$ is satisfied by at most n elements of G for every integer n. Then prove that G is a cyclic group.

Or

(b) State and prove Wedderburn's Theorem on Finite Division Rings.

20. (a) Define the term adjoint in Q. State and prove the properties of adjoint.

Or

(b) Prove that every positive integer can be expressed as the sum of squares of four integers.
