(7 Pages) **Reg. No. : ....**

Code No. : 5332 **Sub. Code : PMAM 43**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2022

Fourth Semester

Mathematics — Core

ADVANCED ALGEBRA — II

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Suppose that L, K, F are there fields in the relation $L \supset K \supset F$. If $[L:F]=20$, $[K:f]=4$ then $[L:K]$ is

   (a) 80 (b) 16

   (c) 5 (d) 24

2. $\left|Q(\sqrt{2}.\sqrt{3}):Q\right|$ is

   (a) 2 (b) 4

   (c) 6 (d) 1

3. The degree of the splitting field of the polynomial x^4+x^2+1 over the field of rational numbers is

   (a) 4 (b) 1

   (c) 2 (d) 24

4. The extension K of F is a simple extension of F if

   (a) K is the splitting field of F

   (b) $K = F(a)$ for some $a \in K$

   (c) $K = F(a)$ for some $\alpha \in F$

   (d) $F = K(\alpha)$ for some $\alpha \in K$

5. Let K be a field and let F be a subfield of K. Then $G(K,F)$ is

   (a) the set of all automorphisms of K

   (b) the set of all automorphimsms of K leaving every element of F fixed

   (c) the set of all automorphisms of F leaving every element of K fixed

   (d) the set of all automorphism of F

6. Let $K = \mathbb{C}$, $F = \mathbb{R}$. The $G(K,F)$ is a group of order

(a) 1 (b) 0

(c) ∞ (d) 2

7. If $\phi_n(x)$ is the n^{th} cyclotonic polynomial then $\phi_1(x)+\phi_2(x)$ is

(a) $2x$ (b) x^2+x+1

(c) $x-1$ (d) 0

8. Let F be a finite field with 25 elements and suppose that $F \subset K$ where K is also a finite field with $[K:F]=5$ then $O(K)$ is

(a) 125 (b) 5

(c) 25^5 (d) 5^{25}

9. If $x = 1-i+j-k$ then $N(x)$ is

(a) 4 (b) 2

(c) 0 (d) $1+i-j+k$

10. "Every positive integer can be expressed as the sum of squares of four integers" – This theorem is known as

(a) Hurwitz theorem

(b) Frobenius theorem

(c) Wedderburn's theorem

(d) Lagrange's theorem

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that "algebraic over algebraic is algebraic".

Or

(b) If $a, b \in K$ are algebraic over F of degree m and n respectively, and if $(m,n)=1$, prove that $F(a,b)$ is of degree mn over F.

12. (a) State and prove the reminder theorem.

Or

(b) If F is of characteristic 0 and $f(x) \in f[x]$ is such that $f'(x)=0$, prove that $f(x) = \alpha \in F$.

13. (a) If K is a field and if $\sigma_1, \sigma_2, .., \sigma_n$ are distinct automorphisms of K, prove that it is impossible to find elements $a_1, a_2, ... a_n$ not all 0, in K such that $a_1\sigma_1(u) + a_2\sigma_2(u) + ... + a_n\sigma_n(u) = 0$ for all $u \in K$.

Or

(b) If $K = C$ and $F = R$, find $G(K, F)$ and the fixed filed of $G(K, F)$.

14. (a) Let F be a finite field with of elements and suppose that $F \subseteq K$ where K is also a finite field. Prove that K has q^n elements where $n = [K : f]$.

Or

(b) For every prime number p and every positive integer m, prove that there exists a filed having p^m elements.

15. (a) Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Prove that $D = C$.

Or

(b) Let Q be the division ring of real quaternions. If $x \in Q$, define $N(x)$ and show that for all $x, y \in Q$, $N(xy) = N(x)N(y)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If L is a finite extension of K and if K is a finite extension of F, prove that L is a finite extension of F and $[L : F] = [L : K][K : F]$.

Or

(b) Prove that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

17. (a) Prove that a polynomial of degree n over a field can have at most n roots in any extension field.

Or

(b) If $\alpha_1, \alpha_2, ..., \alpha_n$ are algebraic over F, prove that there is an element $c \in F(\alpha_1, \alpha_2, ..., \alpha_n)$ such that $F(c) = F(\alpha_1, \alpha_2, ..., \alpha_n)$.

18. (a) If K is the splitting field of some polynomial over F, prove that K is a normal extension of F.

Or

(b) State and prove the fundamental theorem of Galois theory.

19. (a) After proving the necessary lemmas, prove that the multiplicative group of non zero elements of a finite field is cyclic.

Or

(b) Prove that a finite division ring is necessarily a commutative field.

20. (a) State and prove Frofenius theorem.

Or

(b) State and prove the left – division algorithm in the Hurwitz ring of integral quaternions.

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