(8 pages) **Reg. No. :** ....

Code No. : 8357 **Sub. Code : HMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2014.

Second Semester

Mathematics

ALGEBRA — II

(For those who joined in July 2012 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Let V is a finite dimensional vector space over F of dimension n. Then $\dim A(V)$ is

   (a) n (b) n^2

   (c) n^n (d) none of these

2. Let $A : r(ST) \geq \min\{r(T), r(S)\}; B : r(ST) = r(S)$ if S is regular. Then

   (a) Both A and B are true

   (b) A is true and B is false

   (c) B is true and A is false

   (d) Both A and B are false

3. Let V be a finite dimensional vector space and $\{v_1, v_2 .....v_n\}$ be a fixed basis for V. Let T be the identity transformation from V to V. Then the number of zeros in the matrix of T in the basis is

   (a) n (b) n^2

   (c) $n^2 - n$ (d) $n^2 + n$

4. Two nilpotent linear transformation are similar if and only if

   (a) they have same nonzero entries in their matrix

   (b) they have same invariants

   (c) their index of nilpotents are same

   (d) none of these

5. Let $T \in A(V)$. Then $Tr(T)$ is

(a) sum of all characteristic roots

(b) sum of distinct characteristic roots

(c) product of all characteristic roots

(d) product of distinct characteristic root

6. The diagonal elements of a Hermition matrix are

(a) zero (b) real numbers

(c) complex numbers (d) none of these

7. Let $F \subseteq K \subseteq L$ be three fields with $[L:F]=6$, $[L:K]=2$ then $[K:F]$ is

(a) 1 (b) 3

(c) 8 (d) 12

8. Let $f(x) \in F[x]$ be of degree $n \geq 1$. Let E be an extension of F in which $F(x)$ has n roots. Then $[E:F]$ is

(a) n (b) $n!$

(c) at most n (d) at most $n!$

9. Let F be a finite field with q elements and let $F \subset K$ where K is also a finite field. If $[K:F]=n$ then number of elements in K is

(a) nq (b) $q+n$

(c) q^n (d) n^q

10. Consider the statements Let $f(x) \in F[x]$

A: $f(x)$ has multiple roots $\Rightarrow$ $f(x)$ and $f'(x)$ have no common factor

B: $f(x)$ and $f'(x)$ have a nontrivial common factor $\Rightarrow$ $f(x)$ has multiple roots

(a) Both A and B are true

(b) A is true and B is false

(c) B is true and A is false

(d) Both A and B are false

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions choosing either (a) or (b).

11. (a) If V is an n-dimensional vector space over F and $T \in A(V)$ then prove that there exists a non-trival polynomial $q(x) \in F[x]$ of degree atmost n^2, such that $q(T)=0$.

Or

(b) If V is finite dimensional vector space over F then prove that $T \in A(V)$ is regular if and only if T maps V onto V.

12. (a) If $T \in F_n$ and if $F \subset K$, prove that as an element of K_n, T is invertible if and only if it is already invertible in F_n.

Or

(b) If $T \in F_n$ has minimal polynomial $p(x)$, over F, prove that every root of $p(x)$, in its splitting field K, is a characteristic root of T.

13. (a) If λ is a characteristic root of the normal transformation N then prove that $\overline{\lambda}$ is a characteristic root of N^*.

Or

(b) If $T \in A(V)$ then prove that its Hermitian adjoint $T^* \in A(V)$. Also prove that $(T^*)^* = T$.

14. (a) If a, b in K are algebraic over F then prove that $a \pm b, ab$ and $\frac{a}{b}(\text{if } b \neq 0)$ are all algebraic over F.

Or

(b) If L is an algebraic extension of K and if K is an algebraic extension of F then prove that L is an algebraic extension of F.

15. (a) If F is a field of characteristic $p \neq 0$, then prove that the polynomial $x^{p^n} - x \in F[x]$, for $n \geq 1$, has distinct roots.

Or

(b) If p is prime and m is a positive integer then prove that there exists of field having p^m elements.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions choosing either (a) or (b).

16. (a) If $\lambda \in F$ is a characteristic root of $T \in A(V)$ then for any polynomial $q(x) \in F[x]$, prove that $q(\lambda)$ is a characteristic root of $q(T)$ and hence prove that a characteristic root of $T \in A(V)$ is also a root of the minimal polynomial of T.

Or

(b) If V is finite dimensional over F and $T \in A(V)$ then prove that (i) if T is singular then the constant term of the minimal polynomial for T is zero and hence there exists an $S \neq 0$ in $A(V)$ such that $ST = TS = 0$. (ii) if T is right-invertible then prove that it is invertible.

17. (a) If $T \in A(V)$ and $W \subset V$ is invariant under T, then prove that T induces a linear transformation $\overline{T}$ on V/W defined by $(v+W)\overline{T} = vT + W$. If T satisfies the polynomial $q(x) \in F[x]$ then prove that $\overline{T}$ also satisfies $q(x)$. Also prove that if $p_1(x)$ and $p(x)$ are the minimal polynomials of $\overline{T}$ and T respectively then $p_1(x)/p(x)$.

Or

(b) If V is a n dimensional vector space over F and if $T \in A(V)$ has all its characteristic roots in F then prove that T satisfies a polynomial of degree n over F.

18. (a) (i) If $T \in A(V)$ is such that $(vT, v) = 0$ for all $v \in V$ then prove that $T = 0$.

(ii) If $T \in A(V)$ is such that $(vT, vT) = (v, v)$ for all $v \in V$ then prove that T is unitary.

Or

(b) Prove that the linear transformation T in V is unitary if and only if it takes on orthonormal basis of V into an orthonormal basis of V.

19. (a) If $f(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$ then prove that there is an extension E of F, such that $[E : F] \leq n$, in which $f(x)$ has a root.

Or

(b) If F and F' are isomorphic fields and $p(x)$ is irreducible in $F[x]$ and if v is a root of $p(x)$ then prove that $F(v)$ is isomorphic to $F'(w)$ where w is a root of $p'(t)$. Also prove that the above isomorphism σ can be chosen so that (i) $v_\sigma = w$ (ii) $\alpha\sigma = \alpha'$ for every $\alpha \in F$.

20. (a) Suppose $f(x) \in F[x]$ is such that $f'(x) = 0$. Prove that

(i) $f(x) = \alpha \in F$, if F is of characteristic 0.

(ii) $f(x) = g(x^p)$ for some polynomial $g(x) \in F[x]$, if F is of characteristic $p \neq 0$.

Or

(b) If the field F has p^n elements prove that the automorphisms of F form a cyclic group of order n.