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M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2012.

First Semester

Mathematics

ANALYSIS — I

(For those who joined in July 2012 onwards)

Time : Three hours

Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Each questions carries 1 mark.

1. Which of the following is / are true?
 - (a) The set of rational numbers in $[0,1]$ with denominator 2 has a limit point
 - (b) Every finite set is compact
 - (c) The closure of $(a, b]$ and the closure of (a, b) are the same
 - (d) $\frac{1}{2}$ is a limit point of $(0,1)$

2. State Weierstrass theorem.

3. Which of the following is / are true?

- (a) The sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$ is convergent
- (b) Every Cauchy sequence in R^k is convergent
- (c) If $p \leq 1$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p}$ converges
- (d) e is algebraic

4. What is the value of $\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$?

5. Which of the following is / are true?

- (a) Every absolutely convergent series is convergent
- (b) The root test is stronger than the ratio test.
- (c) $a_n \rightarrow 0$ is a necessary condition for $\sum a_n$ to converge
- (d) $a_n \rightarrow 0$ is a sufficient condition for $\sum a_n$ to converge

6. Define rearrangement of series.

7. Which of the following is / are true?

(a) The function f defined by $f(x) = |x|$ is continuous on $\mathbb{R}$

(b) Any function whose domain is a discrete metric space is continuous

(c) The function f defined by

$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$ has a discontinuity of the second kind.

(d) If f and g are monotonic functions, then $f + g$ is also monotonic

8. Give an example of a real valued function on $\mathbb{R}$ which has exactly 2 points of discontinuity.

9. Which of the following is / are true?

(a) Every differentiable function is continuous

(b) The real function defined by $f(x) = |x + 1|$ is differentiable at 0

(c) The real function defined by $f(x) = 2x^4 + 3x^2 + 43$ is differentiable more than 6 times

(d) If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuity

10. State The mean value theorem.

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

Each questions carries 5 marks.

11. (a) Is every point of every nonempty open set $E \subseteq \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in $\mathbb{R}^2$.

Or

(b) Let $K \subset \mathbb{R}^1$ consists of 0 and the numbers $\frac{1}{n}$ for $n = 1, 2, 3, \dots$. Prove that K is compact directly from the definition without using the Heine-Borel theorem.

12. (a) Suppose $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$. Then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Or

(b) Define upper and lower limits of a sequence of real numbers. Give an example of a real sequence with upper limit 4 and lower limit 2.

13. (a) State and prove the ratio test

Or

- (b) For any sequence $\{c_n\}$ of positive numbers prove that $\liminf_{n \rightarrow \infty} \frac{c_n + 1}{c_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{c_n}$ and $\limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \liminf_{n \rightarrow \infty} \frac{c_n + 1}{c_n}$.
14. (a) Prove that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X whenever V is open in Y .
- Or
- (b) Prove that continuous image of a connected space is connected.
15. (a) If f and g are differentiable, then prove that fg is differentiable.

Or

- (b) Suppose f is real and differentiable on $[a, b]$, and suppose that $f'(a) < \lambda < f'(b)$, then prove that there exists an $x \in (a, b)$ such that $f'(x) = \lambda$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

Each question carries 8 marks.

16. (a) Prove that every nonempty perfect set in R^k is uncountable.
- Or
- (b) Prove that every k -cell is compact.
17. (a) Prove that the convergences of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

Or

- (b) If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.
18. (a) Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. The prove that there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that $\liminf_{n \rightarrow \infty} s'_n = \alpha$, $\limsup_{n \rightarrow \infty} s'_n = \beta$.

Or

- (b) (i) Give an example to show that the Cauchy product of two non-absolutely convergent series need not be a convergent series.
- (ii) Give a condition under which the Cauchy product of two convergent series is convergent.
19. (a) (i) Suppose f is a continuous mapping of a compact metric space X into a metric space Y , then prove that f is uniformly continuous on X .
- (ii) Discuss: Can we remove the term "compact" in the above statement?

Or

- (b) Prove that monotonic functions have no discontinuities of the second kind.
20. (a) State and prove L' Hospital rule.

Or

- (b) (i) Prove that the mean value theorem is not true in case of vector valued functions.
- (ii) Prove that the L' Hospital rule is not true in case of vector valued functions.