(8 pages)

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Code No. : 7133 **Sub. Code : PMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2019.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. T is a bounded linear transformation if for every x and $k \geq 0$ s.t

(a) $\|T(x)\| \leq k$ (b) $\|T(x)\| \leq kx$

(c) $\|T(x)\| \leq k\|x\|$ (d) $\|T(x)\| < \infty$

2. $\|x\| - \|y\| - \|x - y\|$

(a) $\leq$ (b) $\geq$

(c) $<$ (d) $>$

3. For every $\mathfrak{C}$ in N^x, $F_{dx}(f) =$ ————.

(a) $F_x(\alpha f)$ (b) $(\alpha F_x)(f)$

(c) $F_x(F(\alpha))$ (d) $(xF_a)(f)$

4. If X is a compact Hausdorff space, than $\mathfrak{C}(X)$ is reflexive if and only if

(a) X is an infinite set

(b) X is a finite set

(c) X is a bounded set

(d) X is not empty

5. If S is a non-empty subset of a Hilbert space than

(a) $S^{\perp\perp} = S^{\perp}$ (b) $S^{\perp\perp} = S^{\perp\perp\perp}$

(c) $S^{\perp} = S^{\perp\perp\perp}$ (d) $S^{\perp\perp\perp\perp} = S^{\perp\perp}$

6. If $\{e_i\}$ is on orthonormal set in a Hilbert space H then $\sum |(x, ei)|^2 \leq \|x\|^2$ for every $x \in H$ is called

(a) Schwarz inequality

(b) Bassel's inequality

(c) Triangle inequality

(d) Spectral inequality

7. Let $\{e_i\}$ be an orthonormal set in a Hilbert space H. Then $\{e_i\}$ is complete is equivalent to

(a) $x \perp \{e_i\} \Rightarrow x \neq 0$

(b) If x is an arbitrary vector is H then $x = \Sigma(x, e_i)\, e_i$

(c) Both (a) and (b) are equivalent

(d) Neither (a) nor (b) is true

8. Let H be a Hilbert space and T^* be adjoint of the operator T which one of the following is true

(a) $(\alpha T)^* = \alpha T^*$ (b) $(\alpha T)^* = \bar{\alpha} T^*$

(c) $(T_1 T_2)^* = T_1{}^* T_2{}^*$ (d) $\|T^* T\| = \|T\|$.

9. If N is a normal operator on H then $\|N^2\| =$

(a) 1 (b) 0

(c) $\|N\|$ (d) $\|N\|^2$

10. If P is a projection on a Hilbert space H. Then one of the following is false

(a) P is a positive operator on H

(b) $\|Px\| \leq \|x\|$ for every $x \in H$

(c) $\|P\| \leq 1$

(d) None of them is true

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then prove that there exists a functional f_0 in N^* such that $f_0(m) = 0$ and $f_0(x) \neq 0$.

Or

(b) Let T be a linear transformation of a normal linear space N into N^*. Prove that T is continuous if and only if it is bounded.

12. (a) If P is a projection on a Banach space B and if M and N are its range and null space, then show that M and N are closed linear subspaces of B such that $B = M \oplus N$.

Or

(b) State and prove closed graph theorem.

13. (a) State and prove the uniform boundedness theorem.

Or

(b) Show that a closed convex set C of a Hilbert space H contains a unique vector of smallest norm.

[P.T.O.]

14. (a) Show that $O^* = O$ and $I^* = I$. Use the later to show that if T is non-singular, then T^* is also non-singular and that in this case $(T^*)^{-1} = (T^{-1})^*$.

Or

(b) Prove that the adjoint operator $T \to T^*$ on $\mathcal{B}(H)$ has the following properties

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $\|T^*T\| = \|T\|^2$.

15. (a) If T is an operator on H then show that T is normal if and only if its real and imaginary parts commute.

Or

(b) If T is normal, then prove that x is on eigen vector of T with eigen value λ if and only if x is an eigen vector of T^* with eigen value $\bar{\lambda}$.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let M be a closed linear subspace of a normed linear space N. Prove that N/M is a normed linear space. Also prove that if N is a Banach space then so is N/M.

Or

(b) State and prove Hahn-Banach theorem.

17. (a) State and prove open mapping theorem.

Or

(b) If N is a normed linear space, then show that the closed unit sphere S^* and N^* is a compact Hausdorff space in the weak * Eopology.

18. (a) If B is a complex Banach space whose norms obeys the parallelogram law and if an inner product is defined by $4 < x, y > = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ then prove that B is a Hilbert space.

Or

(b) Let M and N be closed linear subspaces of a Hilbert space H. If $M \perp N$ then show that the linear sub space $M + N$ is closed and also prove that $H = M \oplus M^{\perp}$.

19. (a) Let H be a Hilbert space and let f be an arbitrary functional in H^*. Then show that there exists a unique vector y in H such that $f(x) = < x, y >$ for every vector x in H.

Or

(b) Prove that the self-adjoint operators in $\mathcal{B}$ (H) from a closed real linear subspace of $\mathcal{B}$ (H) and therefore a real Banach space which contains the identify transformation.

20. (a) Let T be an operator on H and prove the following

(i) T is singular (z) $0 \in \sigma(T)$

(ii) If T is non-singular, then $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$

(iii) If A is non-singular then $\sigma(ATA^{-1}) = \sigma(T)$

(iv) If $T^K = 0$ for some positive integer K, then $\sigma(T) = \{0\}$.

Or

(b) (i) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then show that $N_1 + N_2$ and N_1N_2 are normal.

(ii) An operator T on H is normal if and only if $\|T^* x\| = \|Tx\|$ for every x.
