(6 pages) **Reg. No. :** ....

Code No. : 5833 **Sub. Code : PMAM 13**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2020.

First Semester

Mathematics – Core

ANALYTIC NUMBER THEORY

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If a prime p does not divide a, then $(p, a) =$ —————.

   (a) 1 (b) 0

   (c) – 1 (d) 2

2. If d / n implies —————

   (a) m / d (b) d / m

   (c) ad / an (d) $0 / d$

3. For $n = 1$, the Mobius function $\mu(n) =$ ————.

(a) 1 (b) 0

(c) – 1 (d) 2

4. Dirichlet multiplication is ————.

(a) Commutative (b) Associative

(c) Both (a) and (b) (d) Distributive

5. The identity function $I(n) = \left[\frac{I}{n}\right]$ is ————.

(a) not multiplicative

(b) multiplicative

(c) completely multiplicative

(d) Additive

6. The Euler totient $\varphi(n)$ is ————.

(a) not multiplicative

(b) multiplicative

(c) completely multiplicative

(d) multiplicative but not completely multiplicative

7. $N'(r) = X(1 + \sum \varphi(n)), z \leq n \leq r$ and $x =$ ————————.

(a) 1 (b) 4

(c) 6 (d) 8

8. $\wedge(n)$ has average order ————————

(a) 0 (b) 1

(c) –1 (d) 2

9. For $x \geq 1$, we have $\sum_{n \geq x} \mu(x) \left[\frac{x}{n}\right] =$ ————————

(a) 0 (b) 1

(c) –1 (d) 2

10. $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} =$ ————————.

(a) 0 (b) 1

(c) –1 (d) 2

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that every integer $n > 1$ is either a prime or a product of prime numbers.

Or

(b) Prove that $(ac, bc) = |c|(a, b)$.

12. (a) Prove that for all f we have $I * f = f * I = f$.

Or

(b) If $n \geq 1$, prove that $\wedge(n) = \Sigma_{d/n}\, \mu(d) \log \frac{n}{d} =$

$- \Sigma_{d/n}\, \mu(d) \log d.$

13. (a) If f is multiplicative then prove that

$\Sigma_{d/n}\, \mu(d) f(d) = \prod_{p/n} (1 - f(p)).$

Or

(b) For $n \geq 1$, prove that $\sigma_{\alpha}^{-1}(n) = \Sigma_{d/n}\, d^{\alpha} \mu(d) \mu\left(\frac{n}{d}\right).$

14. (a) If $x \geq 1$, prove that $\Sigma_{n \leq x}\, n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + 0(x^{\alpha})$ if $\alpha \geq 0$.

Or

(b) For all $x \geq 1$, prove that $\Sigma_{n \leq x}\, \varphi(n) = \frac{3}{\pi^2} x^2 + 0(x \log x).$

15. (a) For all $x \geq 1$, prove that $\sum_{n \leq x} \mu(n)\left[\frac{x}{n}\right] = 1$.

Or

(b) For $x \geq 2$, prove that

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt.$$

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) State and prove the Division Algorithm.

Or

(b) Prove that given any two integers a and b, there is a common divisor d of a and b of the form $d = ax + by$, where x and y are integers. Moreover, every common divisor of a and b divides this d.

17. (a) If $n \geq 1$, prove that $\sum_{d \setminus n} \varphi(d) = n$.

Or

(b) If $n \geq 1$, prove that $\varphi(n) = n \prod_{p/n}\left(1 - \frac{1}{p}\right)$.

18. (a) Prove that if f and g are multiplicative, so is their Dirichelet product $f * g$

Or

(b) Let f be multiplicative, then prove that f is completely multiplicative if and only if, $f^{-1}(n) = \mu(n)f(n)$ for all $n \geq 1$.

19. (a) State and prove Euler's Summation formula.

Or

(b) If $x \geq 1$, prove that $\sum_{n \leq x} \frac{1}{n} = \log x + C + 0\left(\frac{1}{x}\right)$.

20. (a) For all $x \geq 2$, prove that $\log[x]! = x \log x - x + 0(\log x)$.

Or

(b) Prove that for $n \geq 2$,

$$\frac{n}{6 \log n} < \pi(n) < \frac{6n}{\log n}.$$