(7 pages) **Reg. No. : ....**

Code No. : 5846 **Sub. Code : PMAM 33**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2020.

Third Semester

Mathematics

ADVANCED ALGEBRA — I

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)
Answer ALL questions.

Choose the correct answer :

1. If $\dim V = 5$ then $\dim \operatorname{Hom}(V, F)$ is

(a) 1 (b) 25

(c) 5 (d) 6

2. If $w = \dfrac{v}{\| v \|}$ then (w, w) is

(a) 0 (b) 1

(c) $\| v \|$ (d) $\dfrac{1}{\| v \|}$

3. Suppose A is an algebra of dimension m over f. Which one of the following holds?

(a) any m elements in A are linearly dependent

(b) any m elements in A are linearly independent

(c) any $(m+1)$ elements in A are linearly dependent

(d) any $(m+1)$ elements in A are linearly independent

4. If $m(S)=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $m(T)=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$ then $m(ST)$ is

(a) $\begin{pmatrix}3 & 6\\ 9 & 12\end{pmatrix}$ (b) $\begin{pmatrix}0 & 0\\ 3 & 8\end{pmatrix}$

(c) $\begin{pmatrix}3 & 6\\ 2 & 3\end{pmatrix}$ (d) $\begin{pmatrix}3 & 6\\ 5 & 12\end{pmatrix}$

5. $S, T \in A(V)$ are said to be similar if there exists an invertible element $C \in A(V)$ such that

(a) $T = SCS^{-1}$ (b) $T = S^{-1}CS$

(c) $T = CSC^{-1}$ (d) $T = CSC$

6. M_3 will denote the 3×3 matrix

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

7. If A' is the transpose of A, which one of the following is not true.

(a) $(A')' = A$ (b) $(A+B)' = A' + B'$

(c) $(AB)' = A'B'$ (d) $(A^{-1})' = (A')^{-1}$

8. The secular equation of $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ is

(a) $x^2 - x - 6$ (b) $x^2 + x - 6$

(c) $x^2 - x + 6$ (d) $x^2 + x + 6$

9. If T is unitary and it λ is a characteristic root of T then

(a) λ is real (b) $|\lambda| = 1$

(c) $\lambda = \pm\ 1$ (d) $\lambda = 0$

10. If $1+i$ is a characteristic root of the normal transformation N and if $VN=(1+i)v$ then $vN+vN^{x}$

(a) $2v$ (b) $2iv$

(c) $(2+2i)v$ (d) v

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) Show that $\text{Hom}(V, W)$ is a vector space under suitable operations.

Or

(b) It W is a subspace of V, define $W^{\perp}$ and show that $W^{\perp}$ is a subspace of V and $W \cap W^{\perp} = 10$.

12. (a) If V is finite dimensional over F, prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.

Or

(b) If $\lambda \in F$ is a characteristic root of $T \in A(V)$, prove that for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

13. (a) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, Prove that T satisfies a polynomial of degree n over F.

Or

(b) If $T \in A(V)$ us nilpotent, prove that $\alpha_0 + \alpha_1 T + ... + \alpha_m T^m$ where the $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

14. (a) For all $A, B \in F_n$, prove that $(AB)' = B'A'$.

Or

(b) Prove that every $A \in F_n$ satisfies its secular equation.

15. (a) Define a Hermitian linear transformation. If $T \in A(V)$ is Hermitian, prove that all its characteristic roots are real.

Or

(b) Define a unitary linear transformation. If T is unitary and if λ is a characteristic root of T, prove that $|\lambda| = 1$

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) For finite dimensional vectors spaces V and W, find the dimension of $\operatorname{Hom}(V, W)$

Or

(b) Let F be the real field and let V be the set of polynomials, in a variable x, over F degree 2 or less. If $p(x), q(x) \in V$, define $(p(x), q(x)) = \int_{-1}^{1} p(x)\, q(x)\, dx$. Find an orthonormal basis from the basis $v_1 = 1, v_2 = x, v_3 = x^2$ of V.

17. (a) Define (i) right invertible (ii) left invertible linear transformations. Give an example of a right invertible, element in $A(V)$ but is not invertible with justification.

Or

(b) If V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1, v_2, \ldots, v_n$ and the matrix $m_2(T)$ in the basis $w_1, w_2 .., w_n$ of V over F, prove that there is an element $C \in F_n$ such that $m_2(T) = cm_1(T)c^{-1}$.

18. (a) With usual notations, prove that for each $i = 1, 2, k$, $V_i \neq (0)$ and $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$. Also find the minimal polynomial of T_i.

Or

(b) Prove that two nilpotent linear transformations are similar if and only if they have the same invariants.

19. (a) For $A, B \in F_n$ and $\lambda \in F$, prove that

(i) $t_r(\lambda A) = \lambda t_r A$

(ii) $t_r(A + B) = t_r A + t_r B$

(iii) $t_r(AB) = t_r(BA)$ and

(iv) $t_r(A \subset A^{-1}) = t_r C$ if A is invertible.

Or

(b) If $A \in F_n$, define det A with an example and show that $\det A = 0$ if two rows of A are equal.

20. (a) If $T \in A(V)$ is such that $(vT, v) = 0$ for all $v \in V$ prove that $T = 0$.

Or

(b) Prove that the linear transformation T on V is unitary it and only if it takes an orthonormal basis of V into an orthonormal basis of V.