(7 pages) **Reg. No. :** ....

Code No. : 6836 **Sub. Code : PMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021.

Second Semester

Mathematics — Core

ALGEBRA — II

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. The number of ideals of the set of all rational numbers is ___________.

   (a) 1 (b) 2

   (c) 0 (d) none of the above

2. Suppose γ is a real number $0 \le \gamma \le 1$, $M_\gamma = \{f(x) \in R \mid f(\gamma) = 0\}$ is a ___________ ideal of R.

   (a) Left ideal (b) Right ideal

   (c) Prime ideal (d) Maximal ideal

3. The number of units in the ring of integers is ___________.

(a) 1 (b) 2

(c) 0 (d) none of the above

4. The gcd of $3 + 4i$ and $4 - 3i$ in $J[i]$ is ___________.

(a) $2 - i$ (b) $2 + i$

(c) $1 + 2i$ (d) none of the above

5. The content of the polynomial $x^6 - 6x + 1$ is ___________.

(a) 0 (b) 1

(c) 2 (d) none of the above

6. Which of the following is the unique factorization domain?

(a) Z (b) $Z\left(\sqrt{-5}\right)$

(c) (a) and (b) (d) no one of the above

7. The only idempotent element is *rad R* is

(a) 0 (b) 1

(c) 2 (d) none of the above

8. Let $F[(x)]$ be the ring of formal power series over a field F. Then $rad\ F[[x]] =$ ___________.

(a) (0)
(b) (1)
(c) (x)
(d) none of the above

9. A ring R is subdirectly irreducible if and only if the heart of R is not equal to ___________.

(a) $\{1\}$
(b) $\{0\}$
(c) R
(d) None of the above

10. If $R^{\wedge} \neq \{0\}$, then the annihilator of the set of zero divisors of R is ___________.

(a) R
(b) $\{0\}$
(c) $R^{\wedge}$
(d) None of the above

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If ϕ is a homomorphism of R into R' with kernel $I(\phi)$, then prove that (i) $I(\phi)$ is a subgroup of R under addition, (ii) If $a \in I(\phi)$ and $r \in R$ then both ar and ra are in $I(\phi)$.

Or

(b) If U is an ideal of the ring R, then prove that R/U is a ring and is a homomorphic image of R.

12. (a) Let R be a Euclidean ring. Then any two elements a and b in R have a greatest common divisor d. Moreover $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$. Prove.

Or

(b) Let R be a Euclidean ring and $a, b \in R$. If $b \neq 0$ is not a unit in R, then $d(a) < d(ab)$.

13. (a) State and prove the Gauss lemma.

Or

(b) Define primitive polynomial and prove that if $f(x)$ and $g(x)$ are primitive polynomials, then $f(x)g(x)$ is a primitive polynomial.

14. (a) Let I be an ideal of R. Then prove that $I \subseteq rad\ R$ if and only if each element of the coset $1 + I$ has an inverse in R.

Or

(b) For any ring R, prove that the quotient ring $R / Rad\ R$ is without prime radical.

15. (a) For any ring R, the J-radical $J(R)$ is an ideal of R.

Or

(b) An element $a \in R$ is quasi-regular if and only if $a \in I_a$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that every integral domain can be imbedded in a field.

Or

(b) Let R and R' be rings and ϕ a homomorphism of R onto R' with kernel U. Then R' is isomorphic to R/U. Moreover there is one-to-one correspondence between the set of ideals of R' and the set of ideals of R which contain U. This correspondence can be achieved by associating with an ideal W' in R; the ideal W in R defined by $W = \{x \in R \mid \phi(x) \in W'\}$. With W so defined, R/W is isomorphic to R'/W'. Prove.

17. (a) Define Euclidean ring and prove that $J[i]$ is an Euclidean ring.

Or

(b) If p is a prime number of the form $4n+1$ then $p = a^2 + b^2$ for some integers a and b.

18. (a) State and prove the Eisenstein criterion.

Or

(b) Define unique factorization domain and prove that if R is a unique factorization domain then so is $R[x_1, x_2, ..., x_n]$.

19. (a) Let I be an ideal of the ring R. Further, assume that the subset $S \subseteq R$ is closed under multiplication and disjoint from I. Then prove that there exits an ideal P which is maximal in the set of ideals which contain I and do not meet S; any such ideal is necessarily prime.

Or

(b) If I is an ideal of the ring R, then (i) $rad(R/I) \supseteq \frac{rad\ R + I}{I}$ and (ii) whenever $I \subseteq rad\ R$, $rad(R/I) = (rad\ R)/I$.

20. (a) A ring R is isomorphic to a subdirect sum of rings R_i if and only if R contains a collection of deals $\{I_i\}$ such that $R/I_i \simeq R_i$ and $\bigcap I_i = \{0\}$.

Or

(b) Let $I_1, I_2, ..., I_n$ be a finite set of ideals of the ring R. If $I_i + I_j = R$ whenever $i \neq j$, then $R/\bigcap I_i \simeq \Sigma \oplus \left(\frac{R}{I_i}\right)$.