(8 pages) **Reg. No. :** ....

Code No. : 5385 **Sub. Code : ZMAM 43**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2024.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2021-2022 only)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. A complete normed linear space is called as ———— space.

   (a) Metric (b) Hilbert

   (c) Empty (d) Banach

2. For a linear transformation T with a real number $k \geq 0$ satisfying $\|Tx\| \leq K\|x\|$ for every x then K is called a ———— for T.

   (a) bound (b) metric

   (c) operator (d) kernal

3. The isometric isomorphism $x \to F_x$ is called the ———— of N into N^{**}.

   (a) bijective (b) injective

   (c) natural imbedding (d) transitive

4. The ———— of the linear transformation T is the subset $B \times B'$ consists of all ordered pairs of the form $(x, T(x))$.

   (a) open (b) graph of T

   (c) open map (d) closed map

5. The set of all vectors orthogonal to a non empty set S is ———— of S.

   (a) orthogonal complement

   (b) perpendicular

   (c) parallel

   (d) equal

6. A complete Banach space whose norm arises from an inner product is said to be ———— space.

   (a) Banach (b) Complete

   (c) Hilbert (d) Lindelof

7. The conjugate operator T^* of T is defined by $(T^* f)x =$ ——————

(a) $T^* f(x)$ (b) $fT^*(x)$

(c) $f(Tx)$ (d) $T^*(f(x))$

8. The value of $\|T^* T\| =$ ——————

(a) $\|T\|^2$ (b) $\|T^*\|$

(c) $\|T\|$ (d) ϕ

9. An operator N on H is said to be —————— if it commutes with its adjoint.

(a) normal (b) singular

(c) bijective (d) orthogonal

10. An operator A on H satisfying the condition $A = A^*$ is called ——————

(a) adjoint (b) self adjoint

(c) inverse (d) unitary

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let M be a linear subspace of a normed linear space N and f be a functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 then prove that f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

Or

(b) Let N and N' be normed linear spaces and T be a linear transformation of N into N' then prove that the following conditions are all equivalent :

(i) T is continuous

(ii) T is continuous at the origin ie $x_n \to 0 \Rightarrow T(x_n) \to 0$

(iii) there exists a real number $k \geq 0$ with the property that $\|Tx\| \leq K\|x\|$ for every $x \in N$.

(iv) if $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N then $T(S)$ in a bounded set in N'.

[P.T.O.]

12. (a) State and prove closed graph theorem.

Or

(b) Let B be a Banach space and M, N be closed linear subspaces of B such that $B = M \oplus N$. If $z = x + y$ is the unique representation of a vector in B as a sum of vectors in M and N then prove that the mapping p defined by $P(z) = x$ is a projection on B whose range and null space are M and N.

13. (a) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

Or

(b) Prove Schwarz inequality.

14. (a) Let H be a Hilbert space and let f be an arbitrary functional in H^{*}. Then prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

Or

(b) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then prove that $\Sigma|(x, e_i)|^2 \leq \|x\|^2$ for every vector x in H.

15. (a) If T is an operator on H for which $(Tx, x) = 0$ for all x, then prove that $T = 0$.

Or

(b) Prove that a closed linear subspace M of H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T^{*}.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$ then prove that N/M is a normed linear space. Also if N is a Banach space so is N/M.

Or

(b) If N and N' are normed linear spaces then prove that the set $\mathcal{B}(N, N')$ of all continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations and norm defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$.

17. (a) Prove that if N is a normed linear space, then the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak*topology.

Or

(b) State and prove open mapping theorem.

18. (a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non zero vector z_0 in H such that $z_0 \perp M$.

Or

(b) State and prove Uniform Boundedness theorem.

19. (a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H then prove that $x - \Sigma(x, e_i)\, e_i \perp e_j$ for each j.

Or

(b) The adjoint operator $T \to T^*$ on $\mathcal{B}(H)$ has the following properties — Prove.

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(\alpha T)^* = \overline{\alpha} T^*$

(iii) $(T_1 T_2)^* = T_2^* T_1^*$

(iv) $\|T^* T\| = \|T\|^2$.

20. (a) If N_1 and N_2 are normal operators on H with either commutes with the adjoint of the other then prove that $N_1 + N_2$ and $N_1 N_2$ are normal.

Or

(b) Prove that if $P_1, P_2, ...P_n$ are the projections on closed linear subspaces $M_1, M_2.....M_n$ of H then $P = P_1 + P_2 ..... + P_n$ is a projection $\Leftrightarrow$ the $P_i s$ are pairwise orthogonal and P is a projection on $M = M_1 + ... + M_n$.
