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Reg. No. : ....

Code No. : 6854 **Sub. Code : PMAM 43**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021.

Fourth Semester

Mathematics — Core

ADVANCED ALGEBRA — II

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. What is the degree of $\sqrt{2}+\sqrt[3]{5}$ over Q?

   (a) 2 (b) 4

   (c) 6 (d) 8

2. The number e is

   (a) rational (b) a unit

   (c) algebraic (d) transcendental

3. If E is the splitting field of $f(x) = x^3 - 2$ over the field of rational numbers then $[E : F]$ is

(a) 3 (b) 6

(c) 2 (d) 4

4. If $f(x) \in F(x)$ is irreducible and if characteristics of F is zero then $f(x)$ has

(a) a unique root (b) more than one root

(c) a multiple root (d) no multiple root

5. With usual notations, $[F(x_1, x_2, ..., x_n) : S] =$

(a) $F(a_1, a_2, ..., a_n)$ (b) S_n

(c) n (d) $n!$

6. If F is the field of real numbers and K is the field of complex numbers then $\circ(G(K, F))$ is

(a) 0 (b) 1

(c) 2 (d) 3

7. The cyclotomic polynomial $\phi_3(x) =$

(a) $x - 1$ (b) $x + 1$

(c) $x^2 + x + 1$ (d) $x^3 + 1$

8. If the field F has p^m elements then the splitting field of $x^{p^m} - x$ has __________ elements.

(a) m (b) p^m

(c) z (d) $p^m - p$

9. Every polynomial of degree n over the field of complex numbers

(a) is irreducible

(b) has only one real root

(c) has all its roots in the field of complex numbers

(d) has no multiple root

10. The irreducible polynomials over the field of real numbers are of degree less than

(a) 3 (b) 4

(c) 2 (d) 7

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If $a, b \in K$ are algebraic over F of degrees m and n respectively, and if m and n are relatively prime, prove that $F(a, b)$ is of degree mn over F.

Or

(b) If
$T = \{\beta_0 + \beta_1 a + ... + \beta_{n-1} a^{n-1} / \beta_o, \beta_1, ,..., \beta_{n-1} \in F\}$
where $a \in K$ is algebraic of degree n, show that $T = F(a)$.

12. (a) If $a \in K$ is a root of $p(x) \in F[x]$, where $F \subset K$, prove that, in $K[x]$, $(x - a) / p(x)$.

Or

(b) If F is a field of characteristic p, show that $x^{p^m} - x \in F[x]$, for $n \geq 1$, has district roots.

13. (a) If K is a finite extension of F, show that $G(K, F)$ is a finite group and $O(G(K, F) \leq [K : F])$.

Or

(b) Prove that $G(K, F)$ is a subgroup of the group of all automorphisms of K.

14. (a) Given F is a finite field with q elements and $F \subset K$ where K is also a finite field. Show that K has q^n elements where $n = [K : F]$.

Or

(b) Show that for every prime number p and every positive integer m there exists a field having p^m elements.

15. (a) Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Prove that D = C.

Or

(b) State and prove Lagrange Identity.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If $a \in K$ is algebraic of degree n over F, show that $[F(a):F] = n$.

Or

(b) If L is a finite extension of K and if K is a finite extension of F, prove that $[L:F] = [L:K][K:F]$.

17. (a) Prove that a finite extension of a field of characteristics D is a simple extension.

Or

(b) If $p(x)$ is a polynomial in $F[x]$ of degree $x \geq 1$ and is irreducible over F, show that there is an extension E of F, such that $[E:F] = n$, in which $p(x)$ has a root.

18. (a) Given $F(x_1, x_2, ..., x_n)$ is the field of rational functions in $x_1, x_2, ..., x_n$ over F. Show that the field S of symmetric rational functions $a_1, a_2, ..., a_n$ is $F(a_1, a_2, ..., a_n)$ and $G(F(x_1, x_2, ..., x_n), S_n) = S$, the symmetric group of degree n.

Or

(b) State and prove the Fundamental Theorem of Galois.

19. (a) Let K be a field and let G be a finite subgroup of the multiplicative group of non zero elements of K. Show that G is a cyclic group.

Or

(b) State and prove Wedderburn theorem on finite division rings.

20. (a) State and prove Left-Division Algorithm in the Hurwitz ring of integral quaternions.

Or

(b) Prove that every positive integer can be expressed as the sum of squares of four integers.