(6 pages) **Reg. No. :** ....

Code No.: 9025 **Sub. Code: PMAO 11**

M.Phil. DEGREE EXAMINATION,
NOVEMBER 2022

First Semester

Mathematics

Elective – BANACH ALGEBRA AND SPECTRAL THEORY

(For those who joined in July 2018-2019 onwards)

Time : Three hours Maximum : 75 marks

PART A — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

1. (a) If ϕ is a complex homomorphism on a complex algebra A with unit e. Prove that $\phi(e)=1$, and $\phi(x)\neq 0$ for every invertible $x\in A$.

Or

(b) Suppose A is a Banach algebra, $x\in G(A)$, $h\in A$, $\|h\|<\frac{1}{2}\left\|x^{-1}\right\|^{-1}$. Prove that $x+h\in G(A)$, and $\left\|(x+h)^{-1}-x^{-1}+x^{-1}hx^{-1}\right\|\leq 2\leq\left\|x^{-1}\right\|^{3}\|h\|^{2}$.

2. (a) Suppose A is a Banach algebra, Ω is open in C, $f\in H(\Omega)$, and f is one-to-one in Ω. Prove that $\tilde{f}$ is a diffeomorphism of A_{Ω} onto $A_{f(\Omega)}$.

Or

(b) Suppose A is a commutative Banach algebra. Prove that

(i) The Gelfand transform is an isometry for every $x\in A$, if and only if $\left\|x^{2}\right\|=\|x\|^{2}$ for every $x\in A$.

(ii) A is semisimple and $\hat{A}$ is closed in $C(\Delta)$ if and only if there exists $k<\infty$ such that $\left\|x^{2}\right\|\leq K\|x\|^{2}$ for every $x\in A$.

3. (a) If the Banach algebra A is commutative and semisimple, then prove that every involution on A is continuous.

Or

(b) Suppose A is a B^{*}-algebra, B is a closed subalgebra of A, $e\in B$, and $x^{*}\in B$ for every $x\in B$. Prove that $\sigma_{A}(x)=\sigma_{B}(x)$ for every $x\in B$.

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4. (a) If $T \in B(H)$ and if $(Tx, x) = 0$ for every $x \in H$. Prove that $T = 0$.

Or

(b) Prove that every nonempty closed convex set $E \subset H$ contains a unique x of minimal norm.

5. (a) Prove that every positive $T \in B(H)$ has a unique positive square root $S \in B(H)$.

Or

(b) If A is a B^*-algebra and if $z \in A$, prove that the positive functional F on A such that $F(e) = 1$ and $F(zz^*) = \|z\|^2$.

PART B — (5 × 10 = 50 marks)

Answer ALL questions, choosing either (a) or (b).

6. (a) Suppose A is a Banach algebra, $x \in A, \|x\| < 1$. Prove that

(i) $e - x$ is invertible,

(ii) $\left\|(e-x)^{-1} - e - x\right\| \le \frac{\|x\|^2}{1-\|x\|}$,

(iii) $|\phi(x)| < 1$ for every complex homomorphism ϕ on A.

Or

(b) Suppose $x \in A_\Omega$ and $f \in H(\Omega)$. Prove that

(i) $\tilde{f}(x)$ is invertible in A if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$.

(ii) $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

7. (a) State and prove the inverse function theorem.

Or

(b) Suppose $f_1, \ldots\ldots f_k \in A(U^n)$, and suppose that to each $x \in \overline{U}^n$ there corresponds at least one i such that $f_i \neq 0$. Prove that the functions $\phi_1, \ldots\ldots \phi_k \in A(U^n)$ such that $f_1(Z)\phi_1(z) + \ldots + f_k(z)\phi_k(z) = 1$..

8. (a) Prove that suppose A is a commutative Banach algebra with an involution, $x \in A$, $x = x^*$ and $\sigma(x)$ contains no real λ with $\lambda \le 0$. Prove that $y \in A$ with $y = y^*$ and $y^2 = x$.

Or

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(b) Every B^*-algebra A has the following properties;

(i) Hermitian elements have real spectra.

(ii) If $x \in A$ is normal, then $\rho(x) = \|x\|$.

(iii) If $y \in A$ then $\rho(yy^*) = \|y\|^2$.

(iv) If $u \in A, v \in A$, $u \geq 0$ and $v \geq 0$, then $u + v \geq 0$.

(v) If $y \in A$, then $yy^* \geq 0$.

(vi) If $y \in A$, then $e + yy^*$ is invertible in A.

9. (a) State and prove Fuglede-Putnam-Rosenblum.

Or

(b) If $T \in B(H)$ and T is normal, Prove that the unique resolution of the identity E on the Borel subsets of $\sigma(T)$ which satisfies

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

10. (a) Suppose $T \in B(H)$. Prove that

(i) $(Tx, x) \geq 0$ for every $x \in H$ if and only if

(ii) $T = T^*$ and $\sigma(T) \subset (0, \infty)$.

Or

(b) If A is a B^*-algebra, prove that an isometric $*$-isomorphism of A onto a closed subalgebra of $B(H)$, where H is a suitably chosen Hilbert space.

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