(8 pages) **Reg. No. :** ....

Code No. : 6837 **Sub. Code : PMAM 22**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021.

Second Semester

Mathematics — Core

ANALYSIS — II

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ then $fg \in$

   (a) $\mathcal{R}^2(\alpha)$

   (b) $\mathcal{R}(\alpha)$

   (c) $\mathcal{R}(\alpha^2)$

   (d) None of these

2. $f \in \mathcal{R}(\alpha)$ if

(a) f is continuous on $[a, b]$

(b) f is monotonic on $[a, b]$

(c) f is bounded on $[a, b]$

(d) none of these

3. $\lim\limits_{m \to \alpha} \lim\limits_{n \to \alpha} (\cos(m!\pi x))^{2n} =$

(a) 0 (b) 1

(c) −1 (d) none of these

4. Let $f_n(x) = n^2 x (1 - x^2)^n \quad (0 \le x \le 1,\ n = 1, 2, 3, \ldots)$. Then $\frac{1}{2}$ is the value of

(a) $\lim\limits_{n \to \alpha} f_n(x)$ (b) $\lim\limits_{n \to \alpha} \int_0^1 f_n(x)dx$

(c) $\int_0^1 \left(\lim\limits_{n \to \alpha} f_n(x)\right) dx$ (d) none of these

5. If $\mathcal{A}$ has the property that $f \in \mathcal{A}$ whenever $f_n \in \mathcal{A}$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathcal{A}$ is said to be

(a) uniformly closed (b) pointwise closed

(c) closed (d) none of these

6. $\int_{-1}^{1}(1-x^2)^n\,dx$ is

(a) less than $\frac{1}{\sqrt{n}}$ (b) equal to $\frac{1}{\sqrt{n}}$

(c) greater than $\frac{1}{\sqrt{n}}$ (d) none of these

7. Let K be compact and let $f_n \in \mathscr{C}(K)$ $n = 1, 2, 3, \ldots$. $\{f_n\}$ contains a uniformly convergent subsequence is ___________.

(a) $\{f_n\}$ is pointwise bounded

(b) $\{f_n\}$ is equicontinuous on K

(c) Both (a) and (b) are true

(d) Neither (a) nor (b) is true

8. Suppose the series $\sum_{0}^{\infty} C_n x^n$ converges for $\|x\| < R$ then $\sum_{1}^{\infty} nC_n x^{n-1}$ converges in

(a) $\left(-\frac{1}{R}, \frac{1}{R}\right)$ (b) $(-2R, 2R)$

(c) $(-R, R)$ (d) None of these

9. $\Gamma\left(\frac{1}{2}\right) =$ __________.

(a) π

(b) $\sqrt{\pi}$

(c) $\sqrt{\frac{\pi}{2}}$

(d) $\frac{\pi}{2}$

10. The sequence of complex functions $\{\phi_n\}$ is said to be orthonormal if

(a) $\int_a^b \phi_n(x)^2 dx = 1$

(b) $\int_a^b \phi_n(x) dx = 1$

(c) $\int_a^b |\phi_n(x)|^2 dx = 1$

(d) $\int_a^b \phi_n^2(x) dx = 1$

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) State and prove fundamental theorem of Calculus.

Or

(b) Prove that $f \in \mathscr{R}(\alpha)$ on [a, b] if and only if for every $\varepsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

12. (a) Prove that the limit of the integral need not be equal to the integral of the limit even if both are finite.

Or

(b) State and prove the Cauchy Criterion for Uniform Convergence.

13. (a) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$, prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

Or

(b) If K is a compact metric space, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \ldots.$ and if $\{f_n\}$ converges uniformly on K then show that $\{f_n\}$ is equicontinuous on K.

14. (a) Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Then show that $\mathcal{B}$ is a uniformly closed algebra.

Or

(b) Suppose ΣC_n converges. Put $f(x) = \sum_{n=0}^{\infty} C_n x^n$ $(-1 < x < 1)$. Then show that $\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} C_n$.

15. (a) If $f(x)=0$ for all x in some segment J then show that $\lim S_N(\rho : x)=0$ for every $x \in J$.

Or

(b) If $x > 0$ and $y > 0$ then show that

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$ and $h(x)=\phi(f(x))$ on $[a, b]$. Then show that $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

Or

(b) Assume α is increased monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then prove that $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$ and $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$.

17. (a) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then show that $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$ $(a \le x \le b)$.

Or

(b) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

18. (a) If γ' is continuous on $[a, b]$ then prove that γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)dt|$.

Or

(b) Let $\{f_n\}$ be a sequence of functions such that $f_n \to f$ uniformly on E in a metric space. Let x be a limit point of E. Then show that $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$.

19. (a) State and prove the Stone-Weierstrass theorem.

Or

(b) Given a double sequence $\{a_{ij}\}$ $(i = 1, 2, 3, ...)$, $(j = 1, 2, 3, ...)$, suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ $(i = 1, 2, 3, ...)$ and Σb_i converges. Then prove that

$$\sum_{i-1}^{\infty} \sum_{j=1}^{\infty} aij = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} aij \, .$$

20. (a) State and prove Parseval's Theorem.

Or

(b) Define gamma function. Prove that if f is a positive function on $(0, \infty)$ such that (i) $f(x+1) = xf(x)$ (ii) $f(1) = 1$ (iii) $\log f$ is convex then show that $f(x) = \Gamma(x)$.