(8 pages)

Reg. No. : ....

Code No. : 7772 **Sub. Code : WMAM 34**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2024.

Third Semester

Mathematics — Core

CALCULUS OF VARIATIONS AND INTEGRAL EQUATIONS

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. A sufficient condition that y be a maximum at x_0 relative to values at neighboring points is
   (a) $\frac{dy}{dx}=0$ at $x_0, \frac{d^2y}{dx^2}>0$
   (b) $\frac{dy}{dx}=0$ at $x_0, \frac{d^2y}{dx^2}<0$
   (c) $\frac{d^2y}{dx^2}=0$ at $x_0, \frac{dy}{dx}>0$
   (d) $\frac{d^2y}{dx^2}=0$ at $x_0, \frac{dy}{dx}<0$

2. Solutions of Euler's equation are known as
   (a) Elementary solutions
   (b) Euler's solutions
   (c) Externals
   (d) Primitive solution

3. The operators δ and $\frac{d}{dx}$ are commutative if
   (a) x is the independent variable
   (b) x is the dependent variable
   (c) y is the dependent variable
   (d) y is the independent variable

4. The condition $\left[F+(g'-y')\frac{\partial F}{\partial y'}\right]_{x=x_2}=0$ is called
   (a) Transversality condition
   (b) Euler's condition
   (c) Normalizing condition
   (d) Rayleigh's condition

5. The degrees of freedom for the compound pendulum is
   (a) 1 (b) 2
   (c) 3 (d) 4

6. The energy difference $L=T-V$ is called
   (a) Kinetic energy (b) Potential energy
   (c) Euler's function (d) the kinetic potential

7. An integral equation is an equation

(a) involving integers

(b) in which a function to be determined appears under an integral sign

(c) involving derivative of functions

(d) in which a function to be determined appears as a difference of two functions

8. The differential operator Z is given by

(a) $\frac{d}{dx}\left(p\frac{d}{dx}\right)+q$ (b) $\frac{d}{dx}(p)+\frac{d}{dx}(q)$

(c) $\frac{d}{dx}\left(p\frac{d}{dx}\right)$ (d) $p\frac{d^2}{dx^2}+\frac{dp}{dx}+q$

9. If $I(y)=\int_0^1\sqrt{1+y^{1^2}}\,dx$, then $I(x)$ is

(a) 2 (b) 0

(c) $\sqrt{2}$ (d) 1

10. The function $G(x, \varepsilon)$ which represents the effect at x due to a unit concentrated cause at ε is often known as the

(a) influence function (b) distribution function

(c) deflection function (d) separable function

11. If $F=1+x+y+y^{1^2}$, $y=\sin x$, $dx=\varepsilon$ then dF for $x=0$ is

(a) 0 (b) 2ε

(c) 3ε (d) ε

12. Which one of the following is not a separable kernal?

(a) $\sin(x+\varepsilon)$ (b) $e^x+\sin x+\cos\varepsilon$

(c) $\cos x\tan\varepsilon+x^2\sin\varepsilon$ (d) $e^x\sin\varepsilon+e^\varepsilon\cos x$

13. The characteristic number of a Fredholm equation with a real symmetric Kernel are

(a) all real (b) all purely imaginary

(c) zero (d) unit modulus

14. The integral operator H is defined by the equation

(a) $Hf(x)=\int_a^b f(\varepsilon)d\varepsilon$

(b) $Hf(x)=\int_a^b K(x, \varepsilon)f(\varepsilon)d\varepsilon$

(c) $Hf(x)=\int_a^b K(x, \varepsilon)f(n)d\varepsilon$

(d) $Hf(x)=\frac{1}{2\pi}\int_a^b K(x, \varepsilon)f(\varepsilon)d\varepsilon$

15. The Kernel $K(x,\varepsilon) = \sin x \cos \varepsilon$ has ——— characteristic numbers associated with $(0, 2\pi)$.

(a) double (b) single

(c) no (d) triple

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

16. (a) Determine the point on the curve of intersection of the surfaces $z = xy + 5$, $x + y + z = 1$ which is nearest the origin.

Or

(b) Show that the integral $I = \int_{x_l}^{x_l} F(x, y, y')dx$ is stationary if and only if its first variation vanishes for every permissible variation δy.

17. (a) Using Hamilton's principle to conservative system derive Lagranges equations.

Or

(b) Derive the equation of motion for a simple pendulum.

18. (a) State the four properties of Green's function G.

Or

(b) Derive the Volterra equation of the second kind from the initial value problem $\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y = f(x)$, $y(a) = y_0$, $y'(a) = y_0^1$.

19. (a) Define the separable Kernel and give an example.

Or

(b) Consider $y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x + \varepsilon)\, y(\varepsilon)\, d\varepsilon$.

Determine the characteristic values of λ and the corresponding characteristic functions.

20. (a) Obtain the resolvent Kernel associated with $K(x, \varepsilon) = x\varepsilon$ in (0, 1).

Or

(b) Prove that the characteristic numbers of a Fredholm equation with a real symmetric Kernel are all real.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

Each answer should not exceed 600 words.

21. (a) If $y(x)$ minimizes to integral $\int_{x_1}^{x_2} F(x, y, y')dx$, prove that it must satisfy the Euler's equation.

Or

(b) Obtain the partial differential equation satisfied by the equation of a minimal surface.

22. (a) Derive the Hamilton's principle in its most general from as applied to the motion of a single particle.

Or

(b) Determine $y(x)$ such that $\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx = 0$ where $y(x_1) = y_1$ and $y(x_2) = g(x_2)$ where x_1 is fixed but x_2 is not pre assigned and $g(x)$ is a given function of x.

23. (a) Show that the integral equation corresponding to the boundary value problem $\frac{d^2y}{dx^2} + \lambda y = 0,\ y(0) = 0,\ y(l) = 0$ is a Fredholm equation of the second find.

Or

(b) Transform the problem $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda x^2 - 1)y = 0 \quad y(0) = 0,\ y(1) = 0$ to the integral equation $y(x) = \lambda \int_0^1 G(x, \varepsilon)\varepsilon y(\varepsilon)d\varepsilon$.

24. (a) Prove that any solution of $y(x) = \lambda \int_0^1 (1 - 3x\varepsilon)\, y(\varepsilon)\, d\varepsilon + F(x)$ can be expressed as the sum of $F(x)$ and some linear combination of the characteristic functions.

Or

(b) Obtain the most general solution of the equation $y(x) = \lambda \int_0^{2\pi} \sin(x + \varepsilon)y(\varepsilon)d\varepsilon + F(x)$ when $F(x) = x$ and when $F(x) = 1$ under the assumption that $\lambda \neq \pm 1/\pi$.

25. (a) Explain the method of solving integral equations of the second kind by a method of successive approximations.

Or

(b) If $y_m(x)$ and $y_n(x)$ are characteristics functions of $y(x) = \lambda \int_a^b K(x, \varepsilon)\, y(\varepsilon)\, d\varepsilon$ corresponding to distinct characteristic numbers, prove that $y_m(x)$ and $y_n(x)$ are orthogonal over the interval (a, b).

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