

Reg. No. :

Code No. : 8823

Sub. Code : KMAM 31

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2017.

Third Semester

Mathematics

MEASURE AND INTEGRATION

(For those who joined in July 2016-2017)

Time : Three hours

Maximum : 75 marks

SECTION A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. E is measurable if
 - (a) If A is any set then $m^*(A) = m^*(A \cap E)$
 - (b) There exists a G_δ set $G \subset E$ such that $m^*(E \setminus G) = 0$
 - (c) For each $\epsilon > 0$ there exists a closed set $F \subset E$ for which $m^*(E \setminus F) < \epsilon$
 - (d) None of these

20. (a) State and prove Vitali convergence theorem.

Or

- (b) For a general measure space. State and prove
 - (i) Integral comparison test
 - (ii) Beppo Levi's lemma.

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2. $m^*(B \sim A) = m^*(B) - m^*(A)$ if
- A is measurable
 - $A \subset B$
 - $m^*(A) < \infty$
 - All of the above is true
3. Let f be a Dirichlet function on $[0, 1]$. Then
- $\int_{[0, 1]} f = 0$
 - $\int_{[0, 1]} f = 1$
 - Integral does not exist on $[0, 1]$
 - None of these
4. In Fatou's lemma, $\{f_n\}$ is a sequence of non-negative measurable functions on E . If $\{f_n\} \rightarrow f$ pointwise a.e. on E then which one of the following is true?
- $\int_E f \leq \liminf \int_E f_n$
 - $\int_E f \geq \limsup \int_E f_n$
 - $\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$
 - None

5. Let f be integrable over $[a, b]$ and $0 < h \leq 1$. Then which of the following is false
- $\text{Diff}_h f = \frac{f(x+h) - f(x)}{h}$
 - $AV_h f(x) = \frac{1}{h} \int_x^{x+h} f$
 - $\int_u^v \text{Diff}_h f(x) dx = AV_h f(v) - AV_h f(u)$
 - None of these
6. Which one of the following is false?
- Lebesgue measure on the real line is complete
 - Lebesgue measure on the real line when restricted to σ -algebra of Borel sets is not complete
 - The difference of two measures, one of which is finite, is a signed measure
 - None of these



7. Two measures γ_1 and γ_2 on $(X, \mathcal{B})$ are said to be mutually singular if there are disjoint measurable sets A and B , $X = A \cup B$ with

- (a) $\gamma_1(A) = 0$ and $\gamma_2(B) = 1$
- (b) $\gamma_1(A) = 0$ and $\gamma_2(B) = 0$
- (c) $\gamma_1(A) \neq 0$ and $\gamma_2(B) \neq 0$
- (d) $\gamma_1(A) = 1$ and $\gamma_2(B) = 1$

8. Which one of the following functions is not of bounded variation?

- (a) f is an increasing function on $[a, b]$
- (b) f is a Lipschitz function on $[a, b]$
- (c) Define f on $[0, 1]$ by

$$f(x) = \begin{cases} x \cos\left(\frac{\pi}{2x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

- (d) None of these

9. Let $(X, \mathcal{B}, \mu)$ be a measure space and f a non-negative measurable function on X for which $\int_X f d\mu < \infty$ then one of them is true

- (a) $\{x \in X / f(x) > 0\}$ is σ -finite
- (b) $\{x \in X / f(x) > 0\}$ is not σ -finite
- (c) $\{x \in X / f(x) > 0\}$ is finite
- (d) None of these

10. Consider a σ -algebra $\mathcal{B} = \{\emptyset, X\}$ then the only measurable functions are

- (a) constant functions
- (b) bounded functions
- (c) continuous functions
- (d) none of these

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let A be any set and $\{E_k\}_{k=1}^n$ a finite disjoint collection of measurable sets. Then show that $m^*\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) = \sum_{k=1}^n m^*(A \cap E_k)$.

Or



- (b) Let f be an extended real valued function on E . If f is measurable on E and $f = g$ a.e. on E , then show that g is measurable on E .

12. (a) Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f , then prove that f is measurable on E .

Or

- (b) State and prove Chebychev's inequality.
13. (a) Let C be a countable subset of (a, b) . Then show that there is an increasing function on (a, b) that is continuous at the points in $(a, b) \setminus C$.

Or

- (b) Prove that A function f on a closed, bounded interval is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

14. (a) State and prove Hahn's lemma.

Or

- (b) Let $\mathcal{G}$ be a collection of subsets of set X and $\mu: \mathcal{G} \rightarrow [a, \infty]$ a set function. Define $\mu^*(\phi) = 0$ and for $E \in \mathcal{G}$, $E \neq \phi$, define $\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k)$ where the infimum is taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{G}$ that cover E . Then the set function $\mu^*: 2^X \rightarrow [0, \infty]$ is an outer measure called the outer measure induced by μ . Prove this.

15. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space and f and g be integrable over X . Then prove the following

- (i) For real numbers α and β ; $\alpha f + \beta g$ is integrable over X and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$$

- (ii) If $f \leq g$ a.e. on X then $\int_X f d\mu \leq \int_X g d\mu$

- (iii) If A and B are disjoint measurable subsets of X , then $\int_{A \cup B} f d\mu =$

$$\int_A f d\mu + \int_B f d\mu.$$

Or



- (b) Let $(X, \mathcal{A}, \mu)$ be a measure space and f a non-negative measurable function on X for which $\int_X f d\mu < \infty$, then prove that f is finite a.e on X and $\{x \in X / f(x) > 0\}$ is σ -finite.

SECTION C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

16. (a) Prove that every interval is measurable.

Or

- (b) Prove that the measurability of E is equivalent to the following two assertions

- (i) for $\epsilon > 0$, there exists an open set $O \supset E$ for which $m^*(O \sim E) < \epsilon$
(ii) there exists a G_δ set $G \supset E$ for which $m(G \sim E) = 0$.

17. (a) State and prove Egoroff's theorem.

Or

- (b) Let f and g be non-negative measurable functions on E . Then $\alpha \geq 0$ and $\beta \geq 0$,
 $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$. Prove this.

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18. (a) State and prove Lebesgue theorem.

Or

- (b) Let the function f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{Diff_h f\}_{0 < h \leq 1}$ is uniformly integrable over $[a, b]$. Prove this.

19. (a) Let μ^* be an outer measure on 2^X . Then the collection of sets that are measurable with respect to μ^* is a σ -algebra. If $\bar{\mu}$ is a restriction of μ^* on $\mathcal{A}$, then $(X, \mathcal{A}, \bar{\mu})$ is a complete measure space. Prove this.

Or

- (b) State and prove Jordan's theorem.

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