

(8 Pages)

Reg. No. :

Code No. : 5315

Sub. Code : PMAM 22/
ZMAM 22

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2022

Second Semester

Mathematics — Core

ANALYSIS — II

(For those who joined in July 2017 onwards)

Time : Three hours

Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let $P = \{0, 0.2, 0.8, 0.9, 1\}$ be a partition of $[0, 1]$ which one of the following is refinement of P
- (a) $\{0, 0.2, 0.7, 0.9, 1\}$
 - (b) $\{0, 0.2, 0.6, 0.8, 0.93, 1\}$
 - (c) $\{0, 0.2, 0.3, 0.4, 0.8, 0.9, 1\}$
 - (d) $\{0, 0.2, 0.6, 0.7, 0.8, 1\}$

2. $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$

- (a) $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for every partition P
- (b) there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$
- (c) there exists a partition P such that $L(P, f, \alpha) - U(P, f, \alpha) < \varepsilon$
- (d) $L(P, f, \alpha) - U(P, f, \alpha) < \varepsilon$ for every partition P

3. Let $s_{m,n} = \frac{m}{m+n}$; $m = 1, 2, 3, \dots$, $n = 1, 2, 3, \dots$, then $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_{m,n}$ is

- (a) 1
- (b) $\frac{1}{2}$
- (c) ∞
- (d) 0

4. A sequence $\{f_n\}$ converges to f w.r.t. the metric of $\mathcal{B}(X)$ if and only if

- (a) $f_n \rightarrow f$ on X
- (b) f is continuous on X
- (c) $f_n \rightarrow f$ uniformly on X
- (d) f_n and f are continuous on X

Page 2

Code No. : 5315



5. If K is compact, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \dots$, and if $\{f_n\}$ is _____ and _____ on K then $\{f_n\}$ is uniformly bounded on K .

- (a) pointwise bounded and continuous
 (b) pointwise bounded and equicontinuous
 (c) continuous and differentiable
 (d) a sequence of continuous and bounded functions

6. Let $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$ ($0 \leq x \leq 1$, $n = 1, 2, 3, \dots$) then $f_n(1/8)$ is

- (a) 0 (b) 1
 (c) ∞ (d) 8

7. The set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$. This statement is known as

- (a) Lagrange theorem
 (b) Weierstrass theorem
 (c) Stone's theorem
 (d) Cauchy's theorem

Page 3 Code No. : 5315

8. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = ?$

- (a) e^x (b) e^n
 (c) e (d) 1

9. The Dirichlet Kernel $D_N(x)$ is also equal to

- (a) $\frac{\sin(N+1)x}{\sin x}$ (b) $\frac{\sin(N+1/2)x}{\sin(x/2)}$
 (c) $\frac{\sin(N+1/2)x}{\sin x}$ (d) $\frac{\sin(N+1)x}{\sin(x/2)}$

10. The value of $\Gamma(1/2)$ is

- (a) $1/2$ (b) 1
 (c) π (d) $\sqrt{\pi}$

PART B — ($5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If P^* is a refinement of P , prove that $L(P, f, \alpha) \leq L(P^*, f, \alpha)$.

Or

- (b) If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$, prove that $f \in R(\alpha)$.

Page 4 Code No. : 5315
 [P.T.O.]



12. (a) Give an example to show that a convergent series of continuous functions may have a discontinuous functions sum.

Or

- (b) State and prove the Weierstrass test for uniform convergence.
13. (a) Let α be monotonically increasing on $[a, b]$. suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, \dots$ and suppose $f_n \rightarrow f$ uniformly on $[a, b]$. Prove that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n d\alpha$.

Or

- (b) If K is compact, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \dots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K , then prove that $\{f_n\}$ is uniformly bounded on K .
14. (a) Define an algebra and the uniform closure of an algebra. Let \mathcal{B} be the uniform closure of an algebra \mathcal{A} of bounded functions. Prove that \mathcal{B} is a uniformly closed algebra.

Or

- (b) Suppose $\sum c_n$ converges. Put

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (-1 < x < 1) \quad \text{prove that}$$

$$\lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

15. (a) Define the Dirichlet Kernel $D_N(x)$ and show

$$\text{that } D_N(x) = \frac{\sin(N+1/2)x}{\sin(x/2)} \quad \text{and}$$

$$s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt.$$

Or

- (b) If $x > 0$ and $y > 0$, prove that

$$\int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$, prove that $f_1 + f_2, cf_1 \in \mathcal{R}(\alpha)$ for every constant c and

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha,$$

$$\int_a^b cf_1 d\alpha = c \int_a^b f_1 d\alpha.$$

Or



- (b) Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Prove that $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$ and show that
- $$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx.$$

17. (a) If γ' is continuous on $[a, b]$, prove that γ is rectifiable and $\wedge(\gamma) = \int_a^b |\gamma'(t)| dt$.

Or

- (b) Suppose $f_n \rightarrow f$ uniformly on a set E in a metric space. Prove that
- $$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t).$$

18. (a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

Or

- (b) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E , prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Page 7

Code No. : 5315

19. (a) If f is a continuous complex function on $[a, b]$, prove that there exists a sequence of polynomials p_n such that $\lim_{n \rightarrow \infty} P_n(x) = f(x)$.

Or

- (b) State and prove Taylor's theorem.

20. (a) Suppose a_0, a_1, \dots, a_n are complex numbers, $n \geq 1$, $a_n \neq 0$. $P(z) = \sum_{k=0}^n a_k z^k$. Prove that $P(z) = 0$ for some complex number z .

Or

- (b) If f is a positive function on $(, \infty)$ such that
(i) $f(x+1) = xf(x)$ (ii) $f(1) = 1$ (iii) $\log f$ is convex, prove that $f(x) = \Gamma(x)$.

Page 8

Code No. : 5315

