(7 pages) **Reg. No. :** ....

Code No. : 9357 **Sub. Code : HMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2015.

Second Semester

Mathematics

ALGEBRA II

(For those who joined in July 2012 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. An element $T \in A(V)$ is called ———————— if there exists an $S \in A(V)$ such that $TS = 1$.

   (a) right-invertible (b) left-invertible

   (c) invertible (d) none of these

2. An element $E \in A(V)$ is called an idempotent if ————————.

   (a) $E^3 = E$ (b) $E^2 = E$

   (c) $E = \overline{E}$ (d) None of these

3. The subspace W of V is invariant under $T \in A(V)$ if ————————.

   (a) $WT \supset W$ (b) $WT \subset W$

   (c) $WT = W$ (d) None of these

4. Let F be a field and let A be a matrix in F_n. If $A = (\alpha_{ij})$, then $trA =$ ————————.

   (a) $\sum_{i,j=1}^{\infty} \alpha_{ij}$ (b) $\sum_{i=1}^{\infty} \alpha_{ii}$

   (c) $\sum_{i=i}^{n} \alpha_{ii}$ (d) None of these

5. If * from F_n into F_n is an adjoint of F_n then $(AB)^* =$ ————————.

   (a) AB (b) BA

   (c) A*B* (d) B*A*

6. The element $0 \neq v \in V$ is called a characteristic vector of T belonging to the characteristic root $\lambda \in F$ if ————————.

   (a) $vT \neq \lambda u$ (b) $vT = \lambda v$

   (c) $vT = v$ (d) $v \neq v$

7. If a complex number λ is a characteristic root of a Hermitian matrix, then λ must be ———.

(a) imaginary (b) real

(c) 0 (d) none of these

8. A complex number is said to be an algebraic number if it is algebraic over the field of ——— numbers.

(a) real (b) complex

(c) rational (d) irrational

9. The element $a \in K$ is said to be algebraic of degree ——— over F if it satisfies a non zero polynomial over F of degree ——— but no non zero polynomial of lower degree.

(a) $n, n-1$ (b) $n-1, n-1$

(c) $n-1, n$ (d) n, n

10. If A is an arbitrary matrix, then ——— is/are symmetric.

(a) AA' (b) $A'A$

(c) Both (d) None of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If V is finite dimensional over F, then prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.

Or

(b) Show that the element $\lambda \in F$ is a characteristics root of $T \in A(V)$ if and only if for some $v \neq 0$ in V, $vT = \lambda v$.

12. (a) Prove that the relation of similarity is an equivalence relation in A(V).

Or

(b) If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F then prove that T statisfies a polynomial of degree n over F.

13. (a) If $T \in A(V)$ then prove that $tr\,T$ is the sum of the characteristic roots of T.

Or

(b) If $T \in A(V)$ is such that $(vT, v) = 0$ for all $v \in V$, then prove that $T = 0$.

14. (a) Prove that the elements in a field K which are algebraic over F form a subfield of K.

Or

(b) State and prove the remainder theorem.

15. (a) If F is of characteristic 0 and $f(x) \in F[x]$ is such that $f'(x) = 0$, then prove that $f(x) = \alpha \in F$.

Or

(b) Prove that $(f(x) + g(x))' = f'(x) + g'(x)$ and $(\alpha f(x))' = \alpha f'(x)$ for $f(x)$, $g(x) \in F[x]$ and $\alpha \in F$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If A is an algebra, with unit element, over F then prove that A is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

Or

(b) If $\lambda \in F$ is a characteristic root of $T \in A(V)$ then prove that λ is a root of the minimal polynomial of T.

17. (a) If $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Show that two nilpotent linear transformations are similar if and only if they have the same invariants.

18. (a) (i) If $(vT, vT) = (v, v)$ for all $v \in V$ then prove that T is unitary.

(ii) Show that the linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.

Or

(b) If $T \in A(V)$, then prove that $T^* \in A(V)$ more over

(i) $(T^*)^* = T$

(ii) $(S + T)^* = S^* + T^*$

(iii) $(\lambda S)^* = \bar{\lambda} S^*$

(iv) $(ST)^* = T^* S^*$ for all $S, T \in A(v)$ and all $\lambda \in F$.

19. (a) If V is a finite dimensional vector space over the field K, and if F is a subfield of K such that $[K:F]$ is finite, show that V is a finite dimensional vector space over F and $\dim_F(V) = (\dim_K(V))([K:F])$.

Or

(b) Let F be the field of rational numbers. Determine the degrees of the splitting field of $x^4 - 2$ over F.

20. (a) Show that the polynomial $f(x) \in F(x)$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a nontrivial common factor.

Or

(b) If F is a finite field and $\alpha \neq 0$, $\beta \neq 0$ are two elements of F then prove that we can find elements a and b in F such that $1 + \alpha a^2 + \beta b^2 = 0$. Also prove that any finite field F has p^m elements, where p is the characteristic of F.
