(7 pages) **Reg. No. :** ....

Code No. : 6852 **Sub. Code : PMAM41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021.

Fourth Semester

Mathematics — Core

FUNCTIONAL ANALYSIS

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. T is a bounded linear transformation if for every x and $k \geq 0$ such that

   (a) $\|T(x)\| \leq k$ (b) $\|T(x)\| \leq kx$

   (c) $\|T(x)\| \leq k\|x\|$ (d) $\|T(x)\| < \infty$

2. Let N be a normed linear space and $x, y \in N$. Then

   (a) $\|x + y\| > 0$ (b) $\|x\| < 0$

   (c) $\|x\| \leq 0$ (d) $\|x + y\| - \|x\| - \|y\| \leq 0$

3. If N is a normed linear space of dimension n then $\mathbb{N}^*$ has dimension

(a) n (b) $n-1$

(c) $n-2$ (d) $n-3$

4. For every ζ in N^*, $F_{\alpha x}(f)$ =

(a) $F_x(\alpha f)$ (b) $(\alpha F_x)(f)$

(c) $F_x(f(\alpha))$ (d) $(xF_\alpha)(f)$

5. A ___________ is a complex Banach space whose normarises from an inner product.

(a) Hilbert space (b) Banach space

(c) Boolean Algebra (d) None of these

6. Let H be a Hilbert Space then $H^\perp$

(a) {0} (b) 1

(c) 0 (d) {1}

7. Let $\{e_i\}$ be a complete orthonormal set in a Hilbert Space H, and let x be an arbitrary vector in H. $\{x, e_i\}$ are called ___________ of x.

(a) Fourier expansion (b) Fourier Coefficient

(c) Parsaval's identity (d) None of these

8. An operator T on H for which $\langle Tx, x \rangle = 0 \ \forall x$ then

(a) $T = \mathrm{I}$ (b) $T = 0$

(c) $T \# \mathrm{I}$ (d) $T \# 0$

9. If P is a projection on M then $x \in M \Leftrightarrow$

(a) $\|Px\| = \|x\|$ (b) $Px = \|x\|$

(c) $\|Px\| = x$ (d) None of these

10. If N is normal and α is any scalar then αN is

(a) self adjoint (b) normal

(c) unitary (d) none of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Let N and N' be normed linear spaces and T be a linear transformation of N into N'. Show that the following conditions are equivalent. (i) T is continuous (ii) T is continuous at the origin. (iii) $\exists$ a real number $k \geq 0$, with the property that $\|Tx\| \leq k\|x\| \forall x \in N$.

Or

(b) Prove that if N is a normed linear space and x_0 is a nonzero vector in N, then there exists functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ & $\|f_0\| = 1$.

12. (a) State and prove closed graph theorem.

Or

(b) Let if P is a projection on a Banach space B and if M and N are its range and null space respectively then prove than M and N are closed linear subspaces of B such that $B = M \oplus N$.

13. (a) State and prove Uniform boundedness theorem.

Or

(b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that $M + N$ is closed.

14. (a) For the adjoint operation $T \to T^*$ on $B(H)$, prove that (i) $\|T^*T\| = \|T\|^2$ (ii) $\|T^*\| = \|T\|$.

Or

(b) Let H be a Hilbert space of f be an arbitrary functional in H^*. Show that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

15. (a) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, show that $N_1 + N_2$ and $N_1 N_2$ are normal.

Or

(b) Show that a closed linear subspace M of H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T^*.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) State and prove Hahn Banach Theorem.

Or

(b) Let M be a closed linear subset of a normed linear space N. Prove that $\frac{N}{M}$ is a normed linear space. If N is a Banach space prove that $\frac{N}{M}$ is also a Banace space.

17. (a) State and prove Open mapping theorem.

Or

(b) If N is a normed linear space then prove that the closed unit sphere $S^* = \{f \in N^* : \|f\| \le 1\}$ in N^* is a compact Hausdorff in the weak * topology.

18. (a) Define a convex set. Prove that a convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Or

(b) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{-1}$.

19. (a) Prove the following with usual notations :

(i) $\|T^*T\| = \|TT^*\| = \|T\|^2$.

(ii) If T is nonsingular, then $(T^*)^{-1} = (T^{-1})^*$

(iii) $(\alpha T_1 + \beta T_2)^* = \overline{\alpha} T_1^* + \overline{\beta} T_2^*$.

Or

(b) State and prove Bessel's inequality.

20. (a) Let T be an operator on H. Prove. (i) If A is nonsingular, then $\sigma(ATA^{-1}) = \sigma(T)$, (ii) If $\lambda \in \sigma(T)$ and P is any polynomial, then $P(\lambda) \in \sigma P(T)$, (iii) If $T^k = 0$ for some positive integer k, then $\sigma(T) = \{0\}$.

Or

(b) If $P_1, P_2, ..., P_n$ are projections on the closed linear spaces $M_1, M_2, ..., M_n$ respectively, then $P = P_1 + P_2 + ... + P_n$ is a projection if and only if P_i's are pairwise orthogonal.

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