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M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2013.

Fourth Semester

Mathematics

FUNCTIONAL ANALYSIS

(For those who joined in July 2008–2011)

Time : Three hours

Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

1. Show that in a normed linear space, norm is a continuous function.
2. Give an example of a reflexive space.
3. If M is a closed subspace of a Hilbert H , prove that $M^\perp$ is a closed linear subspace of H .
4. Let y be an element of a Hilbert space H and let f_y be a function on H such that $f_y(x) = \langle x, y \rangle$. Show that f_y is continuous on H .

5. Show that the adjoint operation $T \rightarrow T^*$ on $B(H)$ has the property $(\alpha T)^* = \bar{\alpha} T^*$.
6. When do we say that a projection on a Hilbert space H is a perpendicular projection?
7. If $T \in B(H)$ is normal and T has an eigen value λ , show that $T - \lambda I$ is normal.
8. Define a regular element in a Banach algebra.
9. Show that $\sigma(x)$ is closed where x is an element of a Banach algebra A .
10. Define an ideal in a Banach algebra.

PART B — (5 × 5 = 25 marks)

Answer ALL questions choosing either (a) or (b).

11. (a) Prove Holder's inequality in a Banach space.

Or

- (b) If P is a projection on a Banach space B , and if M and N are its range and null space, then prove that M and N are closed linear subspace of B such that $B = M \oplus N$.

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12. (a) In a Hilbert space, show that the inner product is jointly continuous.

Or

- (b) If $\{e_i\}$ is an orthonormal set in a Hilbert space. Show that $\{e_i\}$ is complete if and only if $\|x\|^2 = \sum |\langle x, e_i \rangle|^2$, $x \in H$.
13. (a) If N_1 and N_2 are normal operator on a Hilbert space H with the property that either commutes with the adjoint of the other, then prove that $N_1 + N_2$ and $N_1 N_2$ are normal.

Or

- (b) If P and Q are the projections on closed linear subspaces M and N of a Hilbert space H , then show that $M \perp N$ if and only if $PQ = 0 \Leftrightarrow QP = 0$.
14. (a) If T is a normal operator on a Hilbert space H , then prove that each M_i reduces T .

Or

- (b) Prove that the set G of regular elements in a Banach algebra A is an open set and also prove that the set S of singular elements in A is a closed set.

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15. (a) If a Banach algebra A is a division algebra, then show that it equals the set of all scalar multiples of the identity.

Or

- (b) If $1 - xr$ is regular in a Banach algebra A , then show that $1 - rx$ is also regular in A , where r is an element of the radical of A .

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

16. (a) State and prove Hahn-Banach theorem.

Or

- (b) State and prove the uniform boundedness theorem.

17. (a) Let M be a closed linear subspace of a Hilbert space H , let x be a vector not in M , and let d be the distance from x to M . Then show that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

Or

- (b) Let y be a fixed vector in a Hilbert space H and let f_y be a scalar valued function on H defined by the formula $f_y(x) = \langle x, y \rangle \quad \forall x \in H$. Show that f_y is a function in H^* , i.e., f_y is a continuous linear functional on H . Further show that $\|y\| = \|f_y\|$.

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[P.T.O.]



18. (a) Let T be an operator on a Hilbert space H . Show that there exists unique operator T^* on H such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$.

Or

- (b) If $P_1, P_2, \dots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \dots, M_n$ of H , then prove that $P_1 + P_2 + \dots + P_n$ is a projection if and only if the P_i 's are pairwise orthogonal and in this case, P is the projection on $M = M_1 + M_2 + \dots + M_n$.
19. (a) (i) If T is an operator on H , then prove that the eigen values of T constitute a non empty finite subset of the complex plane.
- (ii) If T is non-singular then prove that $\lambda \in \sigma(T) \Leftrightarrow \lambda^{-1} \in \sigma(T^{-1})$.

Or

- (b) Prove that the mapping $x \rightarrow x^{-1}$ of a set G of all regular elements in a Banach algebra onto G is a homeomorphism.

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20. (a) Let A be a Banach algebra and $x \in A$. Prove that $\sigma(x)$ is non empty.

Or

- (b) If I is a proper closed two sided ideal in a Banach algebra A , then prove that the quotient algebra A/I is a Banach algebra.

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