(7 pages) **Reg. No. :** ....

Code No. : 5383 **Sub. Code : ZMAM 41**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2024.

Fourth Semester

Mathematics – Core

ADVANCED ALGEBRA – II

(For those who joined in July 2021 – 2022)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. A complex number is said to be ———— number if it is algebraic over the field of rational numbers.

   (a) real

   (b) imaginary

   (c) algebraic

   (d) extension

2. The element $a \in K$ is said to be algebraic of ———— over F if it satisfies a non zero polynomial over F of degree n but no non zero polynomial of lower degree.

   (a) dimension r (b) degree n

   (c) basis n (d) extension

3. A polynomial of degree n over a field can have ———— roots in any extension field.

   (a) atleast n (b) atmost n

   (c) exactly n (d) less than

4. If E is a minimal extension of the field F in which $f(x)$ has n roots where $n = \deg f(x)$ then E is called ————.

   (a) ring (b) basis

   (c) splitting field (d) normal

5. If G is a group of automorphisms of K, then the fixed field of G is the set of all elements $a \in K$ such that $\sigma(a) =$ ———— for all $a \in G$.

   (a) 1 (b) 0

   (c) a (d) e

6. The automorphism σ of K is in ________ if and only if $\sigma(\alpha)=\alpha$ for every $\alpha \in F$.

(a) $G(K, F)$ (b) $G(K\ K)$

(c) $\varphi(G)$ (d) $G(\varphi)$

7. Any two ________ fields having the same number of elements are isomorphic.

(a) finite (b) infinite

(c) equal (d) fixed

8. For every prime number p and every positive integer m there is a unique field having ________ elements.

(a) p (b) p^{α}

(c) α (d) p^{m}

9. The only irreducible polynomials over the field of real numbers are of degree ________ or ________.

(a) 0, 1 (b) 1, 2

(c) 0, 2 (d) 0, ∞

10. If $x \in Q$ then norm of x is defined by $N(x)=$ ________.

(a) xx^{*} (b) xx

(c) x (d) x^{2}

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If L is an algebraic extension of K and if K is an algebraic extension of F then prove that L is an algebraic extension of F.

Or

(b) If $a \in K$ is of algebraic of degree n over F, then prove that $[F(a) : F]=n$.

12. (a) State and prove remainder theorem.

Or

(b) If $f(x) \in F[x]$ is irreducible then prove that

(i) If the characteristic of F is 0, $f(x)$ has no multiple roots

(ii) If the characteristic of F is $p \neq 0$, $f(x)$ has a multiple root only if it is of the form $f(x)=g(x^{p})$.

13. (a) Let K be the splitting field of $f(x)$ in $F[x]$. Let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$. If the roots of $p(x)$ are $\alpha_1, \alpha_2 \ldots, \alpha_r$ then prove that for each i there exist an automorphism σ_i in $G(K, F)$ such that $\sigma_i(\alpha_1)=\alpha_i$.

Or

(b) Prove that a fixed field of G is a subfield of K.

[P.T.O.]

14. (a) Prove that if the finite field F has p^m elements then the polynomial $x^{p^m} - x$ in $F[x]$ factors in $F[x]$ as $x^{p^m} - x = \prod_{\lambda \in F} (x - \lambda)$.

Or

(b) Prove that if R is a ring in which $px = 0$ for all $x \in R$. Where p is a prime number, then $xT_a^{p^m} = xa^{p^m} - a^{p^m} x$.

15. (a) State and prove Lagrange identity.

Or

(b) Prove that the adjoint in Q satisfies

(i) $x^{**} = x$

(ii) $(\delta x + \gamma y)^* = \delta x^* + \gamma y^*$

(iii) $(xy)^* = y^* x^*$

PART C — $(5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that if L is a finite extension of K and if K is a finite extension of F then L is a finite extension of F. More over $[L : F] = [L : K][K : F]$.

Or

(b) Show that the element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F.

17. (a) Prove that if F is of characteristic 0 and if a, b are algebraic over F, then there exists an element $c \in F(a,b)$ such that $F(a,b) = F(c)$.

Or

(b) If $p(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$, and is irreducible over F then prove that there is an extension E of F such that $[E : F] = n$ in which $p(x)$ has a root.

18. (a) Prove that if K is a finite extension of F, then $G(K, F)$ is a finite group and its order $o(G(K, F))$ satisfies $o(G(K, F)) \leq [K : F]$.

Or

(b) Prove that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

19. (a) Let G be a finite abelian group with the property $x^n = e$ is satisfied by atmost n elements of G for every integer n. Then prove that G is a cyclic group.

Or

(b) State and prove Wedderburn theorem.

20. (a) State and prove Frobenius theorem.

Or

(b) Prove that every positive integer can be expressed as the sum of squares of four integers.
